ON A NON BIRATIONAL INVARIANT e AND QUADRATIC ESTIMATES

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1. Introduction

Let S be a nonsingular rational surface

and D a nonsingular curve on S. (S, D) are called pairs and we study such pairs.

The purpose of Cremonian geometry is the study of birational properties of pairs (S, D).

Suppose that $m \geq a \geq 1$.

Then $P_{m,a}[D] = \dim |mK_S + aD| + 1$ are called mixed plurigenera, which depend on S and D.

Letting Z stand for $K_S + D$, we see

$P_{m,m}[D] = \dim|mZ|+1$, called *logarithmic plurigenera* of $S-D$, from which logarithmic Kodaira dimension is introduced, denoted by $\kappa[D]$.

(Note:In a comic book Golgo 13, Kodaira dimension is mentioned.)

Assume that $\kappa[D] = 2$ and that there exist no (-1) curves E such that $E \cdot D \leq 1$.

Such pairs are proved to be minimal models in the birational geometry of pairs.

Moreover, if $S \neq \mathbf{P^2}$, then there exists a surjective morphism $pr : S \to \mathbf{P^1}$ whose general fibers are $\mathbf{P^1}$. The least mapping degree of $pr|_D : D \to \mathbf{P^1}$ for all such pr,is denoted by σ.

By definition, $P_{1,1}[D] = g$, which is the genus of D, and $\overline{g}$ is defined to be $g - 1$.

If $\sigma > 4$ then $D + 2K_S$ is nef and big;

Furthermore, $P_{2,1}[D] = Z^2 - \overline{g} + 1 = A + 1$, where

$A = Z^2 - \overline{g}$;

If $\sigma > 6$ then $|D + 3K_S| \neq \emptyset$ and

$$P_{3,1}[D] = 3Z^2 + 1 - 7\overline{g} + D^2 = 3A - \alpha + 1 = \Omega - \omega + 1$$

where

$\alpha = 4\overline{g} - D^2$,

$\Omega = (3Z - 2D) \cdot Z = 3Z^2 - 4\overline{g}$ and

$\omega = 3\overline{g} - D^2$.

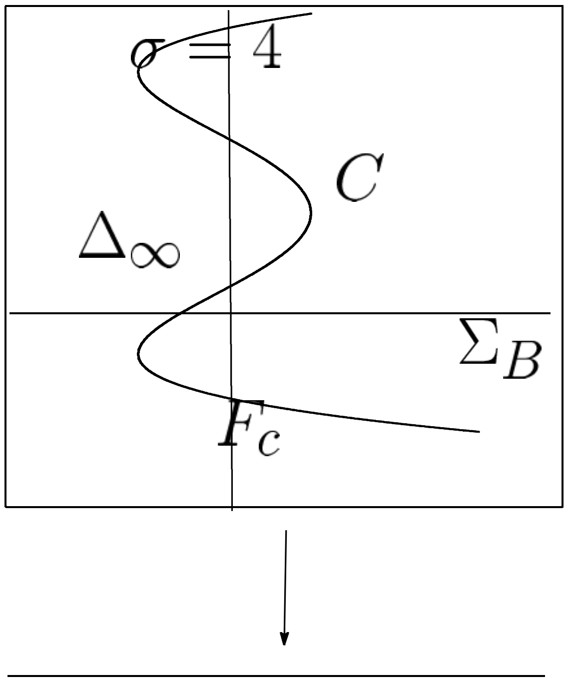
2. PRELIMINARIES

Any nontrivial $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$ has a section Δ_∞ with negative self intersection number, denoted by a symbol Σ_B,
where $-B = {\Delta_\infty}^2$ if $B > 0$.

Σ_B is said to be a Hirzebruch surface of degree B after Kodaira.

Let C be an irreducible curve on Σ_B.
Then $C \sim \sigma\Delta_\infty + eF_c$, for some σ and e.

By $\nu_1, \nu_2, \cdots, \nu_r$ we denote the multiplicities of all singular points (including infinitely near singular points) of C where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.



The symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ is said to be the type of (Σ_B, C).

Definition 1. The pair (Σ_B, C) is said to be **# minimal**, if

- $\sigma \geq 2\nu_1$ and $e - \sigma \geq B\nu_1$;
- moreover, if $B = 1$ and $r = 0$ then assume $e - \sigma > 1$.

2.1. # **minimal pair.**

Theorem 1. *If D is not transformed into a line on $\mathbf{P^2}$ by Cremona transformations, then $\kappa[D] \geq 0$.*

The minimal pair (S, D) is obtained from a # minimal pair (Σ_B, C) by shortest resolution of singularities of C using blowing ups except for $(S, D) = (\mathbf{P^2}, C_d)$, C_d being a nonsingular curve.

Hereafter, suppose that $C \neq \Delta_\infty$. Thus $C \cdot \Delta_\infty = e - B \cdot \sigma \geq 0$ and hence, $e \geq B\sigma$.

Introduce invariants p, u by $\sigma = 2\nu_1 + p, p \geq 0$ and by the following:
(1) $B = 0$. Then $e = \sigma + u$ for some $u \geq 0$.
(2) $B = 1$. Then $e = \sigma + \nu_1 + u$ for some $u \geq 0$.
(3) $B \geq 2$. Then $e = B\sigma + u$ for some $u \geq 0$.

Note :
Given $e > 0$, there are a finite number of types:

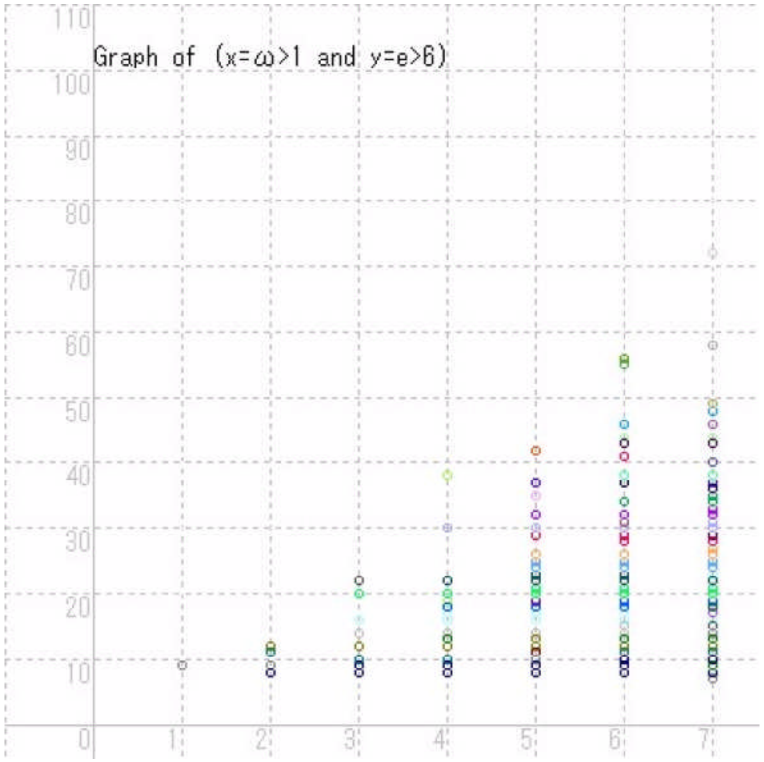


Figure 1

2.2. X and Y.

Proposition 1. *If $B \leq 2$, then letting k denote $wp + 2u$, w being $4 - \delta_{1B}$, we have*

- $X = \sum_{j=1}^{r} \nu_j^2 = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = \sum_{j=1}^{r} \nu_j = 8\nu_1 + k + \omega_1$.

Here $\tilde{k} = kp - 2p^2, \omega_1 = \omega - \overline{g} = K_S \cdot D$.

If $\sigma \geq 6$,then $\sigma \leq (\omega + 1)(\omega + 2)$, $\sigma \leq \omega_1^2 + \omega_1 + 2\overline{g} + 2$.
Here, $\omega_1 = \omega - \overline{g} = D \cdot K_S$.

$$\sigma \leq {A_1}^2 + 3A_1 + 2\overline{g} + 4, \tag{1}$$

$$\sigma \leq (A+2)(A+3). \tag{2}$$

$A = Z^2 - \overline{g}$ and an invariant A_1 is defined to be $A - \overline{g}$, which satisfies

$$A_1 = \frac{(2Z-D)\cdot Z}{2} - \frac{D \cdot Z}{2} = Z \cdot K_S.$$

Introduce invariants $\overline{\nu_j}$ and $\overline{Y}$ by $\overline{\nu_j} = \nu_j - 1$ and $\overline{Y} = \sum_{j=1}^{r} \overline{\nu_j}$. Then $\overline{Y} = Y - r$ and

$$\overline{Y} = 8\overline{\nu_1} + k + A_1.$$

The invariant $\overline{X}$ by

$$\overline{X} = \sum_{j=1}^{r} \overline{\nu_j}^2 = X - 2Y + r,$$

which satisfies that

$$\overline{X} = 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}.$$

Proposition 2. *If $B \leq 2$, then we have*

- $\overline{X} = \sum_{j=1}^{r} \overline{\nu_j}^2 = 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}$
- $\overline{Y} = \sum_{j=1}^{r} \overline{\nu_j} = 8\overline{\nu_1} + k + A_1.$

Defining an invariant $\mathcal{Z}^*$ to be $\overline{\nu_1}\overline{Y} - \overline{X}$, we obtain

$$\mathcal{Z}^* = \sum_{j=2}^{\nu_1 - 1} (\nu_1 - j)(j-1)t_j.$$

Note:

$$\mathcal{Z}^* = (\nu_1 - 2)x_1 + 2(\nu_1 - 3)x_2 + 3(\nu_1 - 4)x_3 + \dots,$$

where $x_1 = t_2 + t_{\nu_1 - 1}, x_2 = t_3 + t_{\nu_1 - 2}, x_3 = t_4 + t_{\nu_1 - 3},.$

Table 1. Yii and Yang

Yii	(陰)	$D^2, \alpha = 4\overline{g} - D^2, \omega = 3\overline{g} - D^2, \omega_1 = \omega - \overline{g}$
Yang	(陽)	$Z^2, A = Z^2 - \overline{g}, \Omega = 3Z^2 - 4\overline{g}, A_1 = A - \overline{g}$
Neutral	(中)	$genus$, $\sigma, Q = (2Z - D)^2, K_S^2$

Table 2. like elementary particles

1st generation	$d, \nu_1, \nu_2, \cdots, \nu_r, \sigma, e, B$
2nd generation	g =genus,
3rd generation	$\alpha, \omega, \alpha_1, \omega_1, A, \Omega, A_1, \Omega_1$ $P_{2,1}[D], P_{3,1}[D], P_{m,a}[D]$,

If $B \leq 2$, then define ε_B to be $1 + \frac{B}{2}$.

[Main Result] (quadratic estimates)

Theorem 2. *Assume $\sigma \geq 7$: If $B \leq 2$, then*

(1) $e \leq \varepsilon_B(\omega + 1)(\omega + 2)$,

(2) $e \leq \varepsilon_B(\alpha + 2)(\alpha + 3)$,

(3) $e \leq \varepsilon_B(A + 2)(A + 3)$,

(4) $e \leq \varepsilon_B(\omega_1^2 + \omega_1 + 2\overline{g} + 2)$,

(5) $e \leq \varepsilon_B(A_1^2 + 3A_1 + 2\overline{g} + 4)$.

Given ω, A, α, e is bounded.

Then there are a finite number of types:

If $e = \varepsilon_B(\omega+1)(\omega+2)$ then

(1) $e = \sigma\varepsilon_B$,

(2) $g = 0, \sigma = (\omega+1)(\omega+2), \omega - 1 = \alpha = A$.

If $B \geq 3$, then

- $e \leq 3\alpha$,
- $e \leq 3\omega$,
- $e \leq 3A$, except for 9 exceptional cases .

If $B \geq 3$, then

- $\sigma \leq 3\omega$,
- $\sigma \leq 3\alpha$.

If $B \leq 2$, then

- $\sigma \leq (\omega + 1)(\omega + 2)$,
- $\sigma \leq (\alpha + 2)(\alpha + 3)$ (By Matsuda),
- $\sigma \leq \omega_1^2 + \omega_1 + 2\overline{g} + 2$,
- $\sigma \leq (A + 2)(A + 3)$,
- $\sigma \leq {A_1}^2 + 2\overline{g} + 3A_1 + 4$,
- $9\sigma \leq {\Omega_1}^2 + 9\Omega_1 + 18\overline{g} + 36$,
- $\sigma \leq \frac{(\Omega+5)(\Omega+8)}{9}$.