On the birational invariants k and (2,1) genus of algebraic plane curves

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1 Introduction

2 basic results

Suppose that (S, D) is a minimal pair with $\kappa[D] = 2$, which is obtained from a # minimal pair(model) (Σ_B, C) by shortest resolution of singularities of C. The type of (Σ_B, C) is denoted by the symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$.

Defining $w = 4 - \delta_{1B}$,we get $w = 4$ if $B \neq 1$. $w = 3$, otherwise.

Proposition 1 *Suppose that $B \leq 2$. Letting k denote $wp + 2u$, we have the following fundamental equalities:*

1. $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
2. $Y = 8\nu_1 + k + \omega_1$.

Here $\tilde{k} = kp - 2p^2, \omega_1 = \omega - \overline{g}$.

2.1 invariant $\widetilde{\mathcal{Z}}$

Following Matsuda([13]), we shall compute $\nu_1 Y - X$, which we denote by $\widetilde{\mathcal{Z}}$. By $\widetilde{\mathcal{Z}} = \nu_1 Y - X = \sum_{j=2}^{\nu_1 - 1}(\nu_1 - j)jt_j \geq 0$, t_j being the number of $j-$ple singular points on C, we have

$$0 \leq \widetilde{\mathcal{Z}} = \nu_1(\omega - \overline{g} - k) - \tilde{k} - \omega_1 + 2\overline{g}. \tag{1}$$

2.2 invariant $\mathcal{Z}^*$

Hereafter suppose that $\nu_1 \geq 3$.

Introducing invariants $\overline{\nu_j}$ and $\overline{Y}$ by $\overline{\nu_j} = \nu_j - 1$ and $\overline{Y} = \sum_{j=1}^{r} \overline{\nu_j}$, respectively, we obtain $\overline{Y} = Y - r$ and

$$\overline{Y} = 8\mu + k + A_1.$$

Moreover, introduce an invariant $\overline{X}$ by

$$\overline{X} = \sum_{j=1}^{r} \overline{\nu_j}^2 = X - 2Y + r,$$

which satisfies that

$$\overline{X} = 8\mu^2 + 2k\mu + \tilde{k} - A_1 - 2\overline{g}.$$

Here, for simplicity, let μ stand for $\overline{\nu_1}$. Thus $\mu \geq 2$.

2.3 case when $B > 2$

However, if $B > 2$, we have fundamental equalities :

1. $\overline{Y} = B_2\sigma + 8\mu + k + A_1$,
2. $\overline{X} = B_2\sigma(\sigma - 2) + 8\mu^2 + 2k\mu + \tilde{k} - A_1 - 2\overline{g}$,

where $B_2 = B - 2$ for $B \geq 2$. Moreover, if $B \leq 2$, we put $B_2 = 0$.

Defining an invariant $\mathcal{Z}^*$ to be $\mu\overline{Y} - \overline{X}$, we obtain

$$-\mu k - \tilde{k} + \nu_1 A_1 + 2\overline{g} - B_2\sigma(\sigma - 2 - \mu) = \mathcal{Z}^*$$

and

$$\mathcal{Z}^* = \sum_{j=2}^{\nu_1 - 1} (\nu_1 - j)(j - 1)t_j.$$

Note:

$$\mathcal{Z}^* = (\mu - 1)y_1 + 2(\mu - 2)y_2 + 3(\mu - 3)y_3 + \ldots \geq 0,$$

where $y_1 = t_2 + t_\mu, y_2 = t_3 + t_{\mu-1}, y_3 = t_4 + t_{\mu-2}, \cdots$.

Moreover, we get

$$B_2\sigma(\sigma - 2 - \mu) \leq -\mu k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

Proposition 2 *If $B \geq 3$, then*

$$\sigma(\sigma - 2 - \mu) \leq \nu_1 A_1 + 2\overline{g} - \mu k - \tilde{k}. \tag{2}$$

Suppose that $p > 0$. Then $\sigma - 2 - \mu \geq 1 + 2(\mu + 1) - 2 - \mu = 1 + \mu$ and $\tilde{k} - k \geq -2$. Hence,

$$\sigma(\mu + 1) \leq \sigma(\sigma - 2 - \mu) \leq \nu_1 A_1 + 2\overline{g} - \mu k - \tilde{k}.$$

However,

$$\nu_1 A_1 + 2\overline{g} - \mu k - \tilde{k} = (\mu + 1)A - (\mu + 1)k + (1 - \mu)\overline{g} - \tilde{q},$$

where $\tilde{q} = \tilde{k} - k \geq -2$.

Therefore

$$\sigma(\mu + 1) \leq (\mu + 1)(A - k) + (1 - \mu)\overline{g} + 2.$$

If $\overline{g} \geq 0$, then $(\sigma - (A - k))(\mu + 1) \leq 2$. Hence, $\sigma \leq A - k$.

If $\overline{g} = -1$, then

$$\sigma(\mu + 1) \leq (\mu + 1)(A - k) + \mu + 1 = (\mu + 1)(A - k + 1)$$

Hence, $\sigma \leq A - k + 1$.

By $\mu \geq 2$, we get $\sigma \geq 7$ and so $k + 6 \leq A$. Thus we obtain the next result.

Proposition 3 *If $B \geq 3$ and $p > 0$, then $\sigma + k - 1 \leq A$.*
In particular, $k + 6 \leq A$.

2.4 example with $B = 3, p = 0$

Suppose that $B = 3, g = 0, p = 0, \mu \geq 2$ and $\mathcal{Z}^* = 0$. Then $\sigma = 2\mu + 2$ and

1. $\overline{Y} = \sigma + 8\mu + k + A + 1$,
2. $\overline{X} = \sigma(\sigma - 2) + 8\mu^2 + 2k\mu - A + 1$.

Therefore, $\mathcal{Z}^* = -2(\mu+1)\mu - k\mu + A(\mu+1) + \mu - 1$ and $\overline{Y} = r\mu$. Hence, $\rho\mu = k + A + 2\mu + 3$. Thus,

$$A = \rho\mu - (k + 2\mu + 3). \tag{3}$$

Moreover, by $\mathcal{Z}^* = 0$ we get

1. $2(\mu+1)\mu = -k\mu + A(\mu+1) + \mu - 1,$
2. $\rho\mu^2 = k\mu + A\mu + 2\mu^2 + 3\mu.$

These imply

$$\rho\mu^2 = A(2\mu+1) + 2\mu - 1 = (A+1)(2\mu+1) - 2.$$

Letting $f = \mu^2, h = 2\mu + 1$, we obtain $1 = 4f + (1-2\mu)h$. Hence, the ideal (f, h) generated by f and h coincides with the ring $\mathbb{Z}[\mu]$. In this case , we say f and h are **relatively prime**.

2.5 a lemma

In general, we have the following result.

Lemma 1 *Let R be a commutative ring. Suppose that there exist elements f and h which are relatively prime. In other words, there exist $u, v \in R$ such that $1 = uf + vh$.*

If $Pf = Qh$ for some $P, Q \in R$,then there exists $L \in R$ such that $P = hL$ and $Q = fL$.

Proof. From $uPf = uQh$, it follows that $uQh = uPf = Puf = P(1 - vh)$. Hence, $P = uQh + vhP$. Thus,$P = hL$, where $L = Qu + Pv$.

2.6 signature of $k - A$

Recalling that $2 = 8f + 2(1-2\mu)h$ and

$$-2 = \rho f - (A+1)h, -2 = -8f + (4\mu - 2)h,$$

we get

$$(\rho + 8)f = (A + 4\mu - 1)h.$$

Since f and h are relatively prime, it follows that $\rho = -8 + hL$, and $A = 1 - 4\mu + fL$ for some L. Then $r = \rho + 8 = hL = (2\mu+1)L$.

By $k = \rho\mu - 2\mu - 3 - A$, we obtain

$$2u = k = -6\mu - 4 + (\mu+1)\mu L.$$

On the other hand, given $\mu \geq 2$ and L, let $\sigma = 2\mu + 2, u = -3\mu - 2 + (\mu+1)\mu L/2$ and $e = 3\sigma + u$. Further, let $r = (2\mu+1)L$ and $\nu_1 = \mu + 1$.

The type becomes $[(2\mu+2) * (6\mu+6+u), 3; (\mu+1)^{(2\mu+1)L}]$. Then $\widetilde{B} = 2e - 3\sigma = 2 + (\mu+1)\mu L$.

The virtual genus g_0 becomes $(2\mu+1)\mu(\mu+1)L/2$. Hence, $g = g_0 - (\mu+1)\mu(2\mu+1)L/2 = 0$.

By definition, $Z_0^2 = (\sigma-2)(\widetilde{B}-4) = -4\mu + 2\mu^2(\mu+1)L$ and $Z^2 = Z_0^2 - r\mu^2 = -4\mu + \mu^2 L$. Hence, $A = 1 - 4\mu + \mu^2 L, k = 2u = -6\mu - 4 + (\mu+1)\mu L$. Then $-k\mu + A(\mu+1) = \mu(2\mu+1) + 1$. Hence, $k - A = \mu L - 2\mu - 5$. The value may be positive or negative.

3 Hartshorne's Lemma

Suggested by Hartshorne ([**?**]), we consider a divisor $2D + \sigma K_S$. The intersection numbers of this and divisor$D$ and Z, will produce useful equalities among invariants.

By $\tilde{\theta_2}$ we denote $(2C + \sigma K_0) \cdot C$. From $(2D + \sigma K_S) \cdot D = 2\sigma\overline{g} - (\sigma-2)D^2$ it follows that

$$2\sigma\overline{g} - (\sigma-2)D^2 = \tilde{\theta_2} + pY + 2\tilde{\mathcal{Z}}.$$

By the way,

$$\begin{aligned}
\tilde{\theta_2} &= (2C + \sigma K_0) \cdot C \\
&= (\sigma Z_0 - (\sigma-2)C) \cdot C \\
&= 2\sigma\overline{g} - (\sigma-2))C^2 \\
&= \sigma(\sigma\widetilde{B} - 2\sigma - \tilde{B}) - (\sigma-2)\sigma\widetilde{B} \\
&= \sigma(\widetilde{B} - 2\sigma).
\end{aligned}$$

If $B \geq 2$ then $\widetilde{B} - 2\sigma = 2u + (B-2)\sigma \geq 0$.
If $B = 0$ then $\widetilde{B} - 2\sigma = 2u \geq 0$.
However, if $B = 1$ then $\widetilde{B} - 2\sigma = 2e - 3\sigma$.
In the case when $2e - 3\sigma \geq 0$, then $\tilde{\theta_2} \geq 0$.

In the case when $2e-3\sigma<0$, letting $L=-(2e-3\sigma)>0$, we consider $\tilde{\theta_3}=(3C+eK_0)\cdot C$. Then

$$\begin{aligned}\tilde{\theta_3}&=(3C+eK_0)\cdot C\\&=(eZ_0-(e-3)C)\cdot C\\&=2e\overline{g}-(e-3))C^2\\&=e(\sigma\widetilde{B}-2\sigma-\widetilde{B})-(e-3)\sigma\widetilde{B}\\&=-2\sigma e-e\widetilde{B}+3\sigma\widetilde{B}\\&=-2\sigma e-2e^2+e\sigma+3\sigma(2e-\sigma)\\&=\sigma(2e-3\sigma)-e(2e-3\sigma)\\&=L(e-\sigma)\\&=L(u+\nu_1).\end{aligned}$$

Thus $\tilde{\theta_3}=L(u+\nu_1)\geq 3L>0$. Moreover,

$$2e\overline{g}-(e-3)D^2=\tilde{\theta_3}+(p+u)Y+3\tilde{\mathcal{Z}}.$$

We say that the type is $\mathrm{RH}_{(+)}$ if either $B\neq 1$ or $B=1$ and $2e-3\sigma\geq 0$. Otherwise, we say the type is $\mathrm{RH}_{(-)}$, namely in the case when $B=1$ and $2e-3\sigma<0$.

Instead of D, we take Z. Then $(2D+\sigma K_S)\cdot Z=\sigma Z^2-2\overline{g}(\sigma-2)$ and by ${\theta_2}^*$ we denote $(2C+\sigma K_0)\cdot Z_0$. Thus

$$\sigma Z^2-2\overline{g}(\sigma-2)={\theta_2}^*+p\overline{Y}+2\mathcal{Z}^*$$

We want to show that ${\theta_2}^*\geq 0$, in the case of $\mathrm{RH}_{(+)}$. Actually,

$$\begin{aligned}{\theta_2}^*&=(2C+\sigma K_0)\cdot Z_0\\&=(\sigma Z_0-(\sigma-2)C)\cdot Z_0\\&=\sigma Z_0^2-2\overline{g_0}(\sigma-2)\\&=\sigma(\sigma\widetilde{B}-4\sigma-2\widetilde{B}+8)-(\sigma-2)(\sigma\widetilde{B}-2\sigma-\widetilde{B}+2)\\&=(\sigma-2)(\widetilde{B}-2\sigma)\end{aligned}$$

Since ${\theta_2}^*=(\sigma-2)(e-B\sigma+2-2\sigma)=(\sigma-2)(2u+B_2\sigma)$, it follows that ${\theta_2}^*=(\sigma-2)(2u+B_2\sigma)\geq 8u\geq 0$, if $B\neq 1$.

If $B=1$ then ${\theta_2}^*=(\sigma-2)(2e-3\sigma)$.

Hence, if $2e - 3\sigma \geq 0$, then ${\theta_2}^* \geq 4(2e - 3\sigma) \geq 0$.

In the case when the type is $\mathrm{RH}_{(-)}$, namely if $2e - 3\sigma = -L$ is negative, consider $(3C + eK_0) \cdot Z_0$, which we denote by ${\theta_3}^*$.

Then ${\theta_3}^* = L(e - \sigma - 1) = L(u + \nu_1 - 1) > 0$ and

$$eZ^2 - 2(e-3)\overline{g} = {\theta_3}^* + (p + u)\overline{Y} + 3\mathcal{Z}^*.$$

By the way,

$$\begin{aligned}
{\theta_3}^* &= (3C + eK_0) \cdot Z_0 \\
&= (eZ_0 - (e-3))C \cdot Z_0 \\
&= eZ_0^2 - 2\overline{g_0}(e-3)) \\
&= e(\sigma\widetilde{B} - 4\sigma - 2\widetilde{B} + 8) - (e-3)(\sigma\widetilde{B} - 2\sigma - \widetilde{B}) \\
&= e(2 + 5\sigma - 2e) - 3\sigma(\sigma + 1) \\
&= e(2 + 2\sigma + L) - 3\sigma(\sigma + 1) \\
&= 2e(\sigma + 1) - 3\sigma(\sigma + 1) + eL \\
&= (\sigma + 1)(2e - 3\sigma) + eL \\
&= L(e - \sigma - 1) \\
&= L(u + \nu_1 - 1) \geq 2L(u + 2) > 0.
\end{aligned}$$

4 Class I.

Class I. We say that the type belongs to Class I, if $p = 0$.

In this class, $\tilde{k} = 0$ and

$$\mathcal{Z}^* = -k\mu + (\mu + 1)A + (1 - \mu)\overline{g}.$$

4.1 case when $\overline{g} \geq 0$

Suppose that $\overline{g} \geq 0$. Then since $\nu_1 \geq 3$ and $\mathcal{Z}^* \geq 0$, it follows that

$$k\mu \leq (\mu + 1)A + (1 - \mu)\overline{g} \leq (\mu + 1)A.$$

Thus

$$k \leq \frac{\mu + 1}{\mu}A.$$

Moreover, assume that $k = \frac{\mu+1}{\mu}A$.

Then $g = 1$ and so $\mathcal{Z}^* = -k\mu + (\mu+1)A = 0$.
Hence, by $\overline{Y} = 8\mu + k + A - \overline{g} = 8\mu + k + A = r\mu$, we obtain

$$\rho\mu = k + A.$$

Here, $\rho = r - 8$. Since $\mu k = (\mu+1)A$, it follows that

$$\rho\mu(\mu+1) = k(\mu+1) + A(\mu+1) = k(2\mu+1). \qquad (4)$$

Letting $f_0 = \mu(\mu+1), h = 2\mu+1$, we obtain $\rho f_0 = kh$. Further, $2f_0 - \mu h = \mu$ and $h - 2\mu = 1 \in J$. Hence, $1 = h - 2(2f_0 - \mu h) = -4f_0 + (1+2\mu)h$. Thus,$f_0$ and h are relatively prime. Moreover, $\rho f_0 = kh$. Therefore, by the previous lemma, $\rho = hL$ and $k = f_0 L$ for some L. Hence, $\rho = (2\mu+1)L$ and $k = \mu(\mu+1)L$.

Conversely, given $\mu \geq 2$ and L, let $\nu_1 = \mu + 1$ and $\sigma = 2\nu_1$.
For $B = 1$, put $e = 3\nu_1 + \mu(\mu+1)L/2$ and $r = 8 + (2\mu+1)L$. The type turns out to be

$$[2(\mu+1) * (3(\mu+1) + \mu(\mu+1)L/2), 1; (\mu+1)^r].$$

Then $u = \mu(\mu+1)L/2, k = 2u$ and $g = 1$.

If $B = 0$, put $e = 2\nu_1 + \mu(\mu+1)L/2$ and $r = 8 + (2\mu+1)L$. The type turns out to be

$$[2(\mu+1) * (2(\mu+1) + \mu(\mu+1)L/2); (\mu+1)^r].$$

Then $u = \mu(\mu+1)L/2, k = 2u, g = 1$.

If $\mu = 2, B = 1$ then the type turns out to be $[6 * (9+3L), 1; 3^{8+5L}]$.

If $\mu = 3, B = 1$ then the type is as follows: $[8 * (12+7L), 1; 3^{8+7L}]$.

Conversely, if the type of the pair is this, then $g = 5\cdot(5+3L) - 3(8+5L) = 1$ and $A = Z^2 = 8\cdot(4+3L) - 4\times(8+5L) = 4L$. $k = 2(3L) = 6L$. Hence, $2k = 3A = 12L$.

4.2 case when $\overline{g} = -1$

Supposing that $g = 0$, we obtain

$$\mathcal{Z}^* = -k\mu + (\mu+1)A + \mu - 1.$$

Then since $\nu_1 \geq 3$ and $\mathcal{Z}^* \geq 0$, it follows that

$$\mu k \leq (\mu + 1)A + \mu - 1. \tag{5}$$

Hence,

$$k \leq \frac{\mu + 1}{\mu} A + 1 - \frac{1}{\mu}.$$

If $k\mu = (\mu + 1)A + \mu - 1$ then $\mathcal{Z}^* = 0$; thus $\overline{Y} = 8\mu + k + A + 1 = r\mu$. Hence,

$$\rho\mu = k + A + 1.$$

Since $k\mu = (\mu + 1)A + \mu - 1$, it follows that

$$\rho\mu(\mu + 1) = k(\mu + 1) + A(\mu + 1) + \mu + 1 = k(2\mu + 1) + 2. \tag{6}$$

Letting $f_0 = \mu(\mu + 1), h = 2\mu + 1$, we obtain

$$\rho f_0 = kh + 2$$

and

$$2f_0 - h\mu = \mu.$$

Hence,

$$1 = h - 2\mu = h - 2(2f_0 - h\mu) = -4f_0 + (1 + 2\mu)h.$$

Combining this with the equation (6), we obtain

$$(\rho + 8)f_0 = (2 + 4\mu + k)h.$$

Therefore, by the previous lemma,

$$r = \rho + 8 = hL, 2 + 4\mu + k = f_0 L$$

for some integer L. Hence,

$$u = \frac{k}{2} = \frac{f_0 L}{2} - 1 - 2\mu.$$

On the other hand, given $\mu \geq 2$ and L, let $\nu_1 = \mu + 1, \sigma = 2\mu + 2, e = 3\nu_1 + u$, where $u = \frac{\mu(\mu+1)L}{2} - 1 - 2\mu$, for some L.

Moreover,in the case when $B = 1$, letting $r = (2\mu + 1)L$, define the type to be $[(2\mu + 2) * (3\mu + 3 + u), 1; (\mu + 1)^{(2\mu+1)L}]$.

Then $\widetilde{B} = 2+\mu(\mu+1)L$ and the virtual genus $g_0 = \mu(\mu+1)(2\mu+1)L/2$. $g = 0$, $k = 2u$ and $Z^2 = \mu^2 L - 4\mu$.

In the case when $B = 0$, letting $r = (2\mu + 1)L$, define the type to be $[(2\mu + 2) * (2\mu + 2 + u); (\mu + 1)^{(2\mu+1)L}]$.

Then $\widetilde{B} = 2+\mu(\mu+1)L$. Furthermore, $g = 0$, $k = 2u$ and $Z^2 = \mu^2 L - 4\mu$.

In both cases, we obtain $A = \mu^2 L - 4\mu + 1$ and $k = 2u = \mu(\mu+1)L - 2 - 4\mu$. Thus $k\mu = (\mu + 1)A + \mu - 1$.

This completes the proof of the following result.

Proposition 4 *Suppose that $p = 0$. Then*

$$k \leq \frac{\mu+1}{\mu} A + 1 - \frac{1}{\mu}.$$

Moreover, $k\mu = (\mu + 1)A + \mu - 1$ if and only if the type is $[(2\mu + 2) * (3\mu + 3 + u), 1; (\mu + 1)^{(2\mu+1)L}]$ or $[(2\mu + 2) * (2\mu + 2 + u); (\mu + 1)^{(2\mu+1)L}]$, where $u = \frac{\mu(\mu+1)L}{2} - 1 - 2\mu$ for some L.

Moreover, if $\mathcal{Z}^* \neq 0$ then $\mathcal{Z}^* \geq \mu - 1$. Hence, $k\mu \leq (\mu + 1)A$.

Proposition 5 *Suppose that $g = 0$ and $p = 0$. If $k\mu < (\mu + 1)A + \mu - 1$, then $k\mu \leq (\mu + 1)A$.*

Furthermore, suppose that $k\mu = (\mu + 1)A$. Then $\mathcal{Z}^* = \mu - 1$, which implies $y_1 = t_2 + t_\mu = 1$. Therefore, we have two cases 1) $t_\mu = 0, t_2 = 1$ and 2) $t_\mu = 1, t_2 = 0$.

Note that

$$\overline{Y} = 8\mu + k + A + 1 = t_2 + (\mu - 1)t_\mu + (r - 1)\mu = 1 + (\mu - 2)t_\mu + (r - 1)\mu.$$

1) If $t_\mu = 0$ then $(\rho - 1)\mu = k + A$. Hence, $(\rho - 1)\mu(\mu + 1) = k(\mu + 1) + A(\mu + 1) = k(2\mu + 1)$.

Since $2\mu + 1$ and $\mu(\mu + 1)$ are relatively prime, it follows that $\rho - 1 = (2\mu + 1)L$ and $k = \mu(\mu + 1)L$ for some L. Thus $r = 9 + (2\mu + 1)L$.

The type turns out to be

$$[2(\mu + 1) * 3(\mu + 1) + \mu(\mu + 1)L/2, 1; 3^r, 2], r = 8 + (2\mu + 1)L.$$

2) If $t_\mu = 1$ then $\mu > 2$ and $(\rho - 1)\mu = k + A - (\mu - 2)$. Thus $\rho\mu = k + A + 2$, and

$$\rho\mu(\mu + 1) = k(\mu + 1) + A(\mu + 1) + 2(\mu + 1) = (1 + k)(2\mu + 1) + 1.$$

For example,if $\mu = 2, B = 1$ then the type turns out to be $[6 * (10 + 3L), 1; 3^{10+5L}]$.

5 Class II.

Class II.

We say that the type belongs to Class II, if $p = 1$.

In this class $p = 1$, $k = w + 2u$ and $\tilde{k} = k - 2$ and

$$\begin{aligned} 0 \le \mathcal{Z}^* &= -k\mu + \nu_1 A + (1-\mu)\overline{g} - \tilde{k} \\ &= -k\mu + \nu_1 A + (1-\mu)\overline{g} + 2 - k \\ &= \nu_1(A-k) + (2-\nu_1)\overline{g} + 2. \end{aligned}$$

Hence,

$$k \le A + \frac{(2-\nu_1)\overline{g} + 2}{\nu_1}. \qquad (7)$$

Thus we obtain the next result.

1. If $g > 0$, then $k \le A$,
2. If $g = 0$, then $k \le A + 1$.

By i we denote $A - k$. Then $i \ge -1$. Moreover,

$$\mathcal{Z}^* = i\nu_1 + (2-\nu_1)\overline{g} + 2 = i(\mu+1) + (1-\mu)\overline{g} + 2. \qquad (8)$$

5.1 case when $i = -1$

Assume that $k = A + 1$. Then $g = 0$ and $\mathcal{Z}^* = 0$. Moreover, $\overline{Y} = 8\mu + k + A + 1 = 8\mu + 2k = r\mu$.

Hence, $\rho\mu = 2k = 2(w + 2u)$, where $w = 3$ or 4.

Assume $B = 1$. Then $w = 3$ and $k = 3 + 2u$. Further, $\sigma = 2\mu + 3, e = 3\mu + 4 + u, r = \rho + 8$

Here are many examples.

For example, if $\rho = 1$, then

$$\mu = 2(3 + 2u) = 6 + 4u.$$

Thus $\nu_1 = 7 + 4u, \sigma = 15 + 8u, e = 22 + 13u$. Therefore, the type becomes $[(15 + 8u) * (22 + 13u), 1; (7 + 4u)^9]$.

If $\rho = 2$, then $r = 10$ and by $\mu = 3 + 2u$, we obtain $\nu_1 = 4 + 2u, \sigma = 9 + 4u, e = 13+7u$. Therefore, the type becomes $[(9+4u)*(13+7u), 1; (4+2u)^{10}]$.

If $\rho = 3$, then $r = 11$ by $3\mu = 6 + 4u$ we get $3(\mu - 2) = 4u$ and so $\mu - 2 = 4L$ and $u = 3L$ for some integer L. Therefore, the type turns out to be $[(7 + 8L) * (10 + 15L), 1; (3 + 4L)^{11}]$.

If $\rho = 4$,then $4\mu = 6 + 4u$, which has no solution.

If $\rho = 5$,then $r = 13$ by $5\mu = 6 + 4u$ we get $5(\mu - 2) = 4(u - 1)$. Thus $\mu = 2 + 4L, u = 1 + 5L$.

Therefore, the type turns out to be $[(7 + 8L) * (11 + 17L), 1; (3 + 4L)^{13}]$. And so on.

5.2 case when $i \geq 0$

If $k \leq A$ then $i = A - k \geq 0$. Moreover,

$$\mathcal{Z}^* = \mu(i - \overline{g}) + i + \overline{g} + 2,$$

and

$$\overline{Y} = 8\mu + 2k + i = 8\mu + 2w + 4u + i.$$

5.3 case when $i = 0$

Suppose that $i = 0$; hence $k = A$ and

$$\mathcal{Z}^* = (1 - \mu)\overline{g} + 2. \tag{9}$$

Thus, $(1 - \mu)\overline{g} + 2 \geq 0$ induces that $g \leq 3$.

5.3.1 case when $g = 0$

Assume that $g = 0$. Then

$$\mathcal{Z}^* = \mu + 1.$$

From $2(\mu - 2) = 2\mu - 4 \leq \mathcal{Z}^* = \mu + 1$, it follows that $\mu \leq 5$.

$\mu = 5$

Suppose that $\mu = 5$.Then $\nu_1 = 6$, $\sigma = 12 + 1 = 13$. Moreover, $\mathcal{Z}^* = 6$ and by definition $\mathcal{Z}^* = 4y_1 + 6y_2$.

From $4y_1 + 6y_2 = 6$, it follows that $y_1 = 0, y_2 = 1$. Hence, $y_2 = t_3 = 1$. Furthermore,

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = (r-1) \cdot \mu + 2.$$

Thus

$$5(\rho - 1) = (\rho - 1) \cdot \mu = 2k - 1. \quad (10)$$

Assume that $B = 1$.

Since $k = 3 + 2u$, we obtain

$$5 \cdot (\rho - 1) = 7 + 4u - 2 = 5 + 4u.$$

Thus $5 \cdot (\rho - 2) = 4u$. Hence, $\rho - 2 = 4L$ and $u = 5L$ for some L. $\rho = 4L + 2$ and $u = 5L$ for some L. Then $r = 10 + 4L$ and so the type becomes $[13 * (19 + 5L), 1; 6^{9+4L}, 3]$.

Assume that $B = 0$. Then the type becomes $[13 * (15 + 5L); 6^{11+4L}, 3]$.

$\mu = 4$

From $\mathcal{Z}^* = \mu + 1 = 5$, it follows that $\mathcal{Z}^* = 3y_1 + 4y_2 = 5$. We have no solution.

$\mu = 3$

Suppose that $\mu = 3$.Then $\nu_1 = 4$, $\sigma = 8 + 1 = 9$. Moreover, $\mathcal{Z}^* = 4$ and by definition $\mathcal{Z}^* = 2y_1$. From $y_1 = 2$, it follows that $y_1 = t_2 + t_3 = 2$. Furthermore,

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = (r-1) \cdot \mu + t_2 + 2t_3.$$

Thus

$$3(\rho - 1) = (\rho - 1) \cdot \mu = 2k + 1 - (t_2 + 2t_3) = 2k - 1 - t_3, \quad (11)$$

where $t_3 = 0, 1, 2$.

Assume that $B = 1$. Then $e = \sigma + 4 + u$. Since $k = 3 + 2u$, we obtain

$$3(\rho - 1) = 5 + 4u - t_3.$$

If $t_3 = 0$ then $3(\rho - 1) = 5 + 4u$. Since $3(\rho - 5) = 4(u - 1)$, it follows that $u = 1 + 3L$ and $\rho = 5 + 4L$ for L.

The type becomes $[9 * (13 + 3L), 1; 4^{11+4L}, 2^2]$.

If $t_3 = 1$ then $3(\rho-2) = 4u+4$. It follows that $u = 3L-1$ and $\rho = 2+4L$ for L.

The type becomes $[9*(12+3L), 1; 4^{8+4L}, 2^2]$.

If $t_3 = 0$ then $3(\rho-2) = 4u+3$. Since $3(\rho-3) = 4u$ it follows that $u = 3L$ and $\rho = 2+4L$ for L.

The type becomes $[9*(13+3L), 1; 4^{9+4L}, 3^2]$.

Assume that $B = 0$. Then $e = \sigma + u$.

Omitted.

$\mu = 2$

Suppose that $\mu = 2$.Then $\nu_1 = 3$, $\sigma = 6+1 = 7$. Moreover, $\mathcal{Z}^* = 2+1 = 3$ and by definition $\mathcal{Z}^* = y_1$. From $y_1 = 2$, it follows that $y_1 = t_2 = 3$. Furthermore,

$$\overline{Y} = 8\cdot\mu + 2k + 1 = (r-3)\cdot\mu + 3.$$

Thus

$$3(\rho-1) = (\rho-1)\cdot\mu = 2k+1-3. \tag{12}$$

Assume that $B = 1$. Then $e = \sigma + 3 + u$. Since $k = 3+2u$, it follows that $2(\rho-3) = 4u+4$.

The type becomes $[7*(10+u), 1; 3^{10+2u}, 2^2]$.

5.3.2 case when $g = 1$

Assume that $g = 1$. Then $k = A$ and $\overline{g} = 1$; hence,

$$\mathcal{Z}^* = 2$$

By $2 = \mathcal{Z}^* \geq \mu - 1$, we obtain $\mu \leq 3$.

$\mu = 3$

Then $2 = \mathcal{Z}^* = 2(t_2+t_3)$, which implies that $t_2 + t_3 = 1$,where $t_3 \leq 1$.
By $\overline{Y} = 8\cdot\mu + 2k = (r-1)\mu + t_2 + 2t_3 = (r-1)\mu + 1 + t_3$, we obtain

$$3(\rho-1) = (\rho-1)\mu = 2k-1-t_3.$$

By $p = 1$ we get $k = w + 2u$.

Suppose that $B = 1$. Then $w = 3$ and

$$3(\rho-1) = (\rho-1)\mu = 2(3+2u) - 1 - t_3 = 5 + 4u - t_3.$$

If $t_3 = 0$ then $3(\rho - 1) = 5 + 4u$; thus $3(\rho - 4) = 4(u - 1)$. Hence, $u = 1 + 3L, \rho = 4 + 4L$.

If $t_3 = 0$ then $3(\rho - 2) = 4u + 3$. Since $3(\rho - 3) = 4u$ it follows that $u = 3L$ and $\rho = 2 + 4L$ for L.

The type becomes $[9 * (14 + 3L), 1; 4^{11+4L}, 2]$.

If $t_3 = 1$ then $3(\rho - 2) = 4u + 4$. Hence, $u = 3L - 1$ and $\rho = 1 + 4L$ for L.

The type becomes $[9 * (12 + 3L), 1; 4^{8+4L}, 3]$.

$\mu = 2$

Then $2 = \mathcal{Z}^* = t_2$, which implies that $t_2 = 2$.

By $\overline{Y} = 8 \cdot \mu + 2k = (r - 2)\mu + 2$, we obtain

$$2(\rho - 2) = (\rho - 2)\mu = 2k - 2.$$

Thus $\rho - 2 = k - 1 = w + 2u - 1$.

If $B = 1$ then $\rho = 3 + 2u + 1$; hence, $r = 12 + 2u$. The type becomes $[7 * (10 + u), 1; 3^{10+2u}, 2^2]$.

5.3.3 case when $g = 2$

Assume that $g = 2$. Then

$$\mathcal{Z}^* = 3 - \mu.$$

By $\mathcal{Z}^* \geq 0$, we get $\mu \leq 3$.

If $\mu = 3$ then $\mathcal{Z}^* = 0$ and $3\rho = 2k - 1$.

It is easy to see that when $B = 1$, the type becomes $[9*(14+3L), 1; 4^{11+4L}]$ for $L \geq 0$.

Similarly, if $\mu = 2$ then $\mathcal{Z}^* = 1$ and $\rho = k$.

It is easy to see that when $B = 1$, the type becomes $[7*(10+u), 1; 3^{10+2u}, 2]$ for $u \geq 0$.

5.3.4 case when $g = 3$

Assume that $g = 3$. Then $\mu = 2$ and $\sigma = 7$. Thus, $\mathcal{Z}^* = (1 - \mu)2 + 2 = 0$. Hence, if $B = 1$ then the type becomes $[7 * (10 + u), 1; 3^{10+2u}]$ for $u \geq 0$.

5.4 case when $i = 1$

Suppose that $A = k + 1$. Then

$$0 \leq \mathcal{Z}^* = \mu(1 - \overline{g}) + 3 + \overline{g}.$$

Thus,

$$\overline{g} \leq \frac{\mu+3}{\mu-1} = 1 + \frac{4}{\mu-1} \leq 5.$$

5.4.1 case when $g = 0$

Assume that $g = 0$. Then

$$\mathcal{Z}^* = 2\mu + 2.$$

By $\mathcal{Z}^* = 2\mu + 2 \geq 3\mu 9$, we get $\mu \leq 11$.

$\mu = 11$

Suppose that $\mu = 11$. Then $\nu_1 = 12$, $\sigma = 25$ and

$$\mathcal{Z}^* = 2\mu + 2 = 24, \mathcal{Z}^* = 10y_1 + 18y_2 + 24y_3 + \cdots .$$

Hence, $y_1 = y_2 = 0, y_3 = t_4 + t_9 = 1$ and we obtain

$$\overline{Y} = 8 \cdot \mu + 2k + 2 = (r-1) \cdot \mu + 3t_4 + 8t_9.$$

Thus,

$$11(\rho - 1) = 2k + 2 - 3 - 5t_9.$$

Assume that $B = 1$. Then $e = \sigma + 12 + u$.

If $t_9 = 1$ then $2k + 2 - 3 - 5t_9 = 8 + 4u - 8 = 4u$. From $11(\rho - 1) = 4u$, it follows that $u = 11L, \rho = 1 + 4L$. Hence, the type becomes $[25 * (37 + 11L), 1; 12^{8+4L}, 9]$.

If $t_4 = 1$ then $2k + 2 - 3 - 5t_9 = 5 + 4u$. From $11(\rho - 1) = 45 + u$, it follows that $u = 7 + 11L, \rho = 4 + 4L$. Hence, the type becomes $[25 * (44 + 11L), 1; 12^{11+4L}, 4]$.

$\mu = 7$

Suppose that $\mu = 7$. Then $\nu_1 = 8$ and $\sigma = 17$.

$$\mathcal{Z}^* = 16, \mathcal{Z}^* = 6y_1 + 10y_2 + 12y_3 + \cdots .$$

Hence, $y_1 = y_2 = 1, y_3 = 0$ and we obtain $t_2 + t_7 = t_3 + t_6 = 1$. Therefore,

$$\overline{Y} = 8 \cdot \mu + 2k + 2 = (r-2)\mu + t_2 + 6t_7 + 2t_3 + 5t_6.$$

Thus

$$7(\rho - 2) = 2k - 1 - 3t_6 - 5t_7 = .$$

Assuming $B = 1$, we get $k = 3 + 2u, e = 17 + 8 + u$. Hence,

$$7(\rho - 2) = 5 + 4u - 3t_6 - 5t_7.$$

If $t_6 = t_7 = 0$ then $7(\rho - 2) = 5 + 4u$. By $7(\rho - 5) = 4(u - 4)$ we obtain $\rho = 5 + 4L, u = 4 + 7L$. Hence, the type becomes $[17 * (25 + 11L), 1; 8^{11+4L}, 3, 2]$.

If $t_6 = 0, t_7 = 1$ then $7(\rho - 2) = 4u$. We obtain $\rho = 2 + 4L, u = 7L$. Hence, the type becomes $[17 * (25 + 11L), 1; 8^{8+4L}, 7, 3]$.

$\mu = 5$

Suppose that $\mu = 5$. Then $\nu_1 = 6$ and $\sigma = 13$.

$$\mathcal{Z}^* = 12, \mathcal{Z}^* = 4y_1 + 6y_2.$$

Hence, 1) $y_1 = 3, y_2 = 0$ or 2) $y_1 = 0, y_2 = 2$.

1). $y_1 = t_2 + t_5 = 3, y_2 = t_3 + t_4 = 0$.

Therefore,

$$\overline{Y} = 8 \cdot \mu + 2k + 2 = (r - 3)\mu + t_2 + 4t_5.$$

Thus

$$5(\rho - 3) = 2k - 1 - 3t_5.$$

Assuming $B = 1$, we get $k = 3 + 2u, e = 13 + 5 + u$. Hence,

$$7(\rho - 2) = 5 + 4u - 3t_5.$$

If $t_5 = 0$ then $t_2 = 3$ and $5(\rho - 2) = 2 + 4u$. By $5(\rho - 5) = 4(u - 2)$ we obtain $\rho = 5 + 4L, u = 2 + 5L$. Hence, the type becomes $[13 * (21 + 5L), 1; 6^{10+4L}, 5, 2^2]$.

If $t_5 = 1$ then $t_2 = 2$ and $5(\rho - 2) = 2 + 4u$. By $5(\rho - 5) = 4(u - 2)$ we obtain $\rho = 5 + 4L, u = 2 + 5L$. Hence, the type becomes $[13 * (21 + 5L), 1; 6^{10+4L}, 5, 2^2]$.

If $t_5 = 2$ then $t_2 = 1$ and $5(\rho - 3) = 4u - 1$. By $5(\rho - 2) = 4(u + 1)$ we obtain $\rho = 2 + 4L, u = 5L - 1$. Hence, the type becomes $[13 * (12 + 5L), 1; 6^{7+4L}, 5^2, 2], L > 0$

If $t_5 = 3$ then $t_2 = 0$ and $5(\rho - 3) = 4u - 4$. Hence, the type becomes $[13 * (20 + 5L), 1; 6^{8+4L}, 5^3]$.

$\mu = 4$

Suppose that $\mu = 4$. Then $\nu_1 = 5$ and $\sigma = 11$.

$$\mathcal{Z}^* = 10, \mathcal{Z}^* = 3y_1 + 4y_2.$$

Then $y_1 = 2, y_2 = 1$.
Hence, the type becomes $[11 * (16 + u), 1; 5^{9+u}, 3, 2^2]$.

$\mu = 3$
Suppose that $\mu = 3$. Then $\nu_1 = 4$ and $\sigma = 9$.

$$\mathcal{Z}^* = 8, \mathcal{Z}^* = 2y_1.$$

$\mu = 2$
Suppose that $\mu = 2$. Then $\nu_1 = 3$ and $\sigma = 7$.

$$\mathcal{Z}^* = 6, \mathcal{Z}^* = y_1 = t_2.$$

Hence, the type becomes $[7 * (10 + u), 1; 3^{9+2u}, 2^6]$.

5.4.2 case when $g = 1$

Assume that $g = 1$.

$$\mathcal{Z}^* = \mu + 3.$$

By $\mu + 3 = \mathcal{Z}^* \geq 2\mu - 4$, we obtain $\mu \leq 7$.

$\mu = 7$
Then $\mathcal{Z}^* = 10 = 6y_1 + 10y_2, y_2 = t_3 + t_6 = 1$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 2t_3 + 5t_6 + (r - 1)\mu = 2 + 3t_6 + (r - 1)\mu.$$

Hence,

$$7(\rho - 1) = 2k - 1 - 3t_6 = 2(3 + 2u) - 1 - 3t_6 = 5 + 4u - 3t_6. \tag{13}$$

We have the following two cases:
1) $t_6 = 0$.
Then $t_3 = 1$ and $7(\rho - 4) = 4(u - 4)$. Thus, $\rho - 4 = 4L, u - 4 = 7L$.
Therefore, the type becomes $[17 * (29 + 7L), 1; 8^{11+4L}, 3]$.

2) $t_6 = 1$.

Then $t_3 = 0$ and $7(\rho - 3) = 4(u - 3)$. Thus, $\rho - 3 = 4L, u - 3 = 7L$.
Therefore, the type becomes $[17 * (28 + 7L), 1; 8^{10+4L}, 6]$.

$\mu = 6$
Then $\mathcal{Z}^* = 9 = 5y_1 + 8y_2 + 9y_3, y_3 = t_4 = 1$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 3 + (r - 1)\mu.$$

Hence,

$$6(\rho - 1) = \mu(\rho - 1) = 2k - 2 = 4(u + 1). \tag{14}$$

Therefore, $\rho - 1 = 2L, u + 1 = 3L$ for some L.
The type becomes $[15 * (21 + 3L), 1; 7^{8+2L}, 4]$.

$\mu = 5$
Then $\mathcal{Z}^* = 8 = 4y_1 + 6y_2, y_1 = t_2 + t_5 = 2$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 2 + 3t_5 + (r - 2)\mu.$$

Hence,

$$5(\rho - 2) = \mu(\rho - 2) = 5 + 4u - 3t_5. \tag{15}$$

$t_5 = 0$
Then $t_2 = 2$ and $5(\rho - 3) = 4u$. Thus, $\rho - 3 = 4L, u = 5L$.
Therefore, the type becomes $[13 * (19 + 5L), 1; 6^{9+4L}, 2^2]$.

$t_5 = 1$
Then $t_2 = 1$ and $5(\rho - 2) = 2 + 4u$. Thus, $\rho - 4 = 4L, u = 2 + 5L$.
Therefore, the type becomes $[13 * (21 + 5L), 1; 6^{10+4L}, 5, 2]$.

$t_5 = 2$
Then $t_2 = 0$ and $5(\rho - 2) = 4u - 1$. Thus, $\rho - 5 = 4L, u = 4 + 5L$.
Therefore, the type becomes $[13 * (23 + 5L), 1; 6^{11+4L}, 5^2]$.

$\mu = 4$
Then $\mathcal{Z}^* = 7 = 3y_1 + 4y_2, y_1 = t_2 + t_4 = y_2 = t_3 = 1$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 3 + 2t_4 + (r - 2)\mu = 3.$$

Hence,

$$4(\rho - 2) = \mu(\rho - 2) = 2k - 2 - 2t_4 = 4 + 4u - 2t_4. \tag{16}$$

$t_4 = 0$
Then $t_2 = 1$ and $\rho - 2 = u + 1$. Thus, $\rho = u + 11, e = 16 + u$.
Therefore, the type becomes $[11 * (16 + u), 1; 5^{u+9}, 3, 2]$.
The case when$t_4 = 1$ does not occur.

$\mu = 3$
Then $\mathcal{Z}^* = 6 = 2y_1, y_1 = t_2 + t_3 = 3$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = t_2 + 2t_3 + (r - 3)\mu = 3 + t_3 + (r - 3)\mu.$$

Hence,

$$3(\rho - 3) = \mu(\rho - 3) = 2k - 2 - t_3 = 2(3 + 2u) - 2 - t_3 = 4 + 4u - t_3. \quad (17)$$

$t_3 = 0$. Then $3(\rho - 3) = 4(u + 1)$. Thus, $\rho - 3 = 4L, u = 3L - 1$.
Therefore, the type becomes $[9 * (12 + 3L), 1; 4^{8+4L}, 2^3]$.

$t_3 = 1$. Then $t_2 = 2$, $3(\rho - 4) = 4u$. Thus, $\rho - 4 = 4L, u = 3L$.
Therefore, the type becomes $[9 * (13 + 3L), 1; 4^{9+4L}, 3, 2^2]$.

$t_3 = 2$. Then $t_2 = 1$, $3(\rho - 5) = 4(u - 1)$. Thus, $\rho - 5 = 4L, u = 1 + 3L$.
Therefore, the type becomes $[9 * (14 + 3L), 1; 4^{10+4L}, 3^2, 2]$.

$t_3 = 3$. Then $t_2 = 0$, $3(\rho - 6) = 4(u - 2)$. Thus, $\rho - 6 = 4L, u = 2 + 3L$.
Therefore, the type becomes $[9 * (15 + 3L), 1; 4^{11+4L}, 3^3]$.

$\mu = 2$
Then $\mathcal{Z}^* = 5 = y_1, y_1 = t_2 = 5$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 5 + (r - 5)\mu.$$

Hence, $r = 14 + 2u, e = 10 + u$.
Therefore, the type becomes $[7 * (10 + u), 1; 3^{9+2u}, 2^5]$.

5.4.3 case when $g = 2$

Assume that $g = 2$. Then

$$\mathcal{Z}^* = 4.$$

By $\mathcal{Z}^* = 4 \geq \mu - 1$, we get $\mu \leq 5$.

$\mu = 5$
Then $\nu_1 = 6$ and $\mathcal{Z}^* = 4 = 4y_1 + 6y_2, y_1 = t_2 + t_5 = 1$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 - 1 = 1 + 3t_5 + (r-1)\mu.$$

Hence,

$$5(\rho - 1) = \mu(\rho - 1) = 2k - 1 - 3t_5 = 5 + 4u - 3t_5. \tag{18}$$

$t_5 = 0$
Then $\nu_1 = 6$ and $5(\rho - 1) = 5 + 4u$. Therefore, $\rho - 3 = 4L, u = 2 + 5L$.
In this case, the type becomes $[13 * (19 + 2 + 5L), 1; 6^{10+4L}, 5]$.

$t_5 = 1$
Then $\nu_1 = 6$ and $5(\rho - 1) = 2 + 4u$. Therefore, $\rho - 2 = 4L, u = 5L$.
In this case, the type becomes $[13 * (19 + 5L), 1; 6^{9+4L}]$.

$\mu = 4$
Then $\nu_1 = 5$ and $\mathcal{Z}^* = 4 = 3y_1 + 4y_2, y_2 = t_3 = 1$ and

$$\overline{Y} = 8 \cdot \mu + 2k + 1 - 1 = 2 + (r-1)\mu.$$

Hence,

$$5(\rho - 1) = \mu(\rho - 1) = 2 + (r-1)u. \tag{19}$$

5.4.4 case when $g = 3$

Assume that $g = 3$. Then

$$\mathcal{Z}^* = 5 - \mu.$$

When $\mathcal{Z}^* = 5 - \mu = 0$, $\mu = 5, \nu_1 = 6$ and $\overline{Y} = 8 \cdot \mu + 2k + 1 - 2 = r \cdot \mu$.
The type becomes $[13 * (19 + 5L), 1; 6^{9+4L}]$.
Conversely, if the pair has this type, $g = 3, A = 4 * 10L, k = 3 + 10L$.

When $\mathcal{Z}^* > 0$,$5 - \mu \geq \mu - 1$. Hence $\mu \leq 3$.

$\mu = 3$.
By $\mathcal{Z}^* = 2 = 2y_1 = 2(t_2+t_3)$, we get $t_2+t_3 = 1$ and $3(\rho-1) = 4+4u-t_3$.

$t_3 = 0$. Then $\sigma = 9, e = 12+3L, r = 9+4L$. The type becomes $[9*(12+5L), 1; 4^{8+4L}, 2]$.

$t_3 = 1$. Then $\sigma = 9, e = 13 + 3L, r = 9 + 4L$. When $B = 1$,the type becomes $[9 * (13 + 3L), 1; 4^{8+4L}, 3]$.

$\mu = 2$.
By $\mathcal{Z}^* = 3 = t_2$, we get $t_2 = 3$ and $2(\rho - 3) = 2k - 4$.
When $B = 1$,the type becomes $[7 * (10 + u), 1; 3^{9+2u}, 2^3]$.

5.4.5 case when $g = 4$

Assume that $g = 4$. Then $\mathcal{Z}^* = -2\mu + 6$. Hence, 1) $\mu = 3$ or 2) $\mu = 2$.

1) $\mu = 3$. Then $\nu_1 = 4$ and $\mathcal{Z}^* = 0$.
Moreover, $\overline{Y} = 8 \cdot \mu + 2k + 1 - 3 = r \cdot \mu$. Hence, $\rho\mu = 2k - 2$. If $B = 1$ then $3\rho = 4(1 + u)$.
Thus, $\rho = 4L, 1 + u = 3L$, for some L. The type becomes $[9 * (12 + 3L), 1, [4^(8 + 4L)]]$.

2) $\mu = 2$. Then $\nu_1 = 3$ and $\mathcal{Z}^* = 2 = t_2$.
Moreover, $\overline{Y} = 8 \cdot \mu + 2k + 1 - 3 = (r - 2) \cdot \mu + 2$. Hence, $(\rho - 2)\mu = 2k - 4$. Thus $\rho = k$. If $B = 1$ then $\rho = k = 3 + 2u$. The type becomes $[7 * (10 + u), 1, [3^(9 + 2u), 2^2]]$.

5.4.6 case when $g = 5$

Assume that $g = 5$. Then $\mu = 2$ and $\mathcal{Z}^* = 1$.
Moreover, $\overline{Y} = 8 \cdot \mu + 2k + 1 - 4 = (r - 1) \cdot \mu + 1$.
Hence, $(\rho - 1)\mu = 2k - 4$. If $B = 1$ then the type becomes $[7 * (10 + u), 1, [3^(9 + 2 * u), 2]]$.

5.4.7 case when $g = 6$

Assume that $g = 6$. Then $\mu = 2$ and $\mathcal{Z}^* = 0$.
Moreover, $\overline{Y} = 8 \cdot \mu + 2k + 1 - 5 = r \cdot \mu$. Hence, $\rho\mu = 2k - 4$. If $B = 1$ then the type becomes $[7 * (10 + u), 1, [3^(9 + 2 * u)]]$.

6 Class IIIa.

Class IIIa. We say that the type belongs to Class IIIa, if $p = 2$ and $B = 1, u = 0$.

Then $k = 6, \tilde{k} = 4$ and

$$\mathcal{Z}^* = \nu_1(A - k) + (1 - \mu)\overline{g} + 2. \tag{20}$$

Thus we obtain the next result.

1. If $g > 0$, then $k \leq A$,
2. if $g = 0$, then $k \leq A + 1$.

6.1 case when $i = -1$

In the case when $k = A + 1$, we get $g = 0$ and

$$\mathcal{Z}^* = \nu_1(A - k) + (1 - \mu)\overline{g} + 2 = -\nu_1 - (1 - \mu) + 2 = 0.$$

From $\overline{Y} = 8\mu + k + A - \overline{g} = 8\mu + 2k$ and $\overline{Y} = r\mu$, it follows that $\rho\mu = 2k = 12$.

Table 1:

ρ	r	μ	ν_1	σ	e	${\nu_1}^r$
1	9	12	13	28	41	13^9
2	10	6	7	16	23	7^{10}
3	11	4	5	12	17	5^{11}
4	12	3	4	10	14	4^{12}
6	14	2	3	8	11	3^{14}

The type becomes

1. $[28 * 41, 1; 13^9]$,
2. $[16 * 23, 1; 7^{10}]$,
3. $[12 * 17, 1; 5^{11}]$,
4. $[10 * 14, 1; 4^{12}]$,
5. $[8 * 11, 1; 3^{14}]$.

If the pairs have these types, then $k = A - 1$.

6.2 case when $i = 0$

Suppose that $k = A$. Then

$$0 \leq \mathcal{Z}^* = (1 - \mu)\overline{g} + 2.$$

Thus $\overline{g} \leq \frac{2}{\mu - 1} \leq 2$.

6.2.1 case when $g = 0$

Assume that $g = 0$. Then

$$\mathcal{Z}^* = \mu + 1.$$

By $2\mu - 4 \leq \mu + 1$, we get $\mu \leq 5$

$\mu = 5$.
$\mathcal{Z}^* = 6$ and $\mathcal{Z}^* = 4y_1 + 6y_2$. Hence, $y_2 = 1$. Thus $t_3 + t_4 = y_2 = 1$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = (r - 1) \cdot \mu + 2t_3 + 3t_4.$$

Accordingly, $5(\rho-1) = 13-2t_3-3t_4$. Thus $t_4 = 1, t_3 = 0$. the type becomes $[14*20, 1, 6^{10}, 4]$.

If $\mu = 4$ then $\mathcal{Z}^* = 5, \mathcal{Z}^* = 3y_1 + y_2$, which has no solution.

$\mu = 3$.

Then

$$\mathcal{Z}^* = 4, \mathcal{Z}^* = 2y_1.$$

Hence, $y_1 = t_2 + t_3 = 2$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = (r-2) \cdot \mu + t_2 + 2t_3.$$

Accordingly, $3(\rho - 2) = 13 - t_2 - 2t_3$. Thus $\rho - 2 = 3$ and the type becomes $[10 * 14, 1; 4^{11}, 3^2]$.

$\mu = 2$.

Then

$$\mathcal{Z}^* = 3, \mathcal{Z}^* = y_1 = t_2.$$

Hence, $t_2 = 3$, $\rho - 3 = 5$ and the type becomes $[8 * 11, 1; 3^{13}, 2^3]$.

6.2.2 case when $g = 1$

Assume that $g = 1$. Then $\mathcal{Z}^* = 2$.

By $\mu - 1 \leq \mathcal{Z}^* = 2$, we get $\mu \leq 3$.

$\mu = 3$.

Then $\mathcal{Z}^* = 2y_1 = 2(t_2 + t_3)$. Hence, $t_2 + t_3 = 1$ and

$$\overline{Y} = 8 \cdot \mu + 2k = (r-1) \cdot \mu + t_2 + 2t_3.$$

Since

$$(\rho - 1) \cdot \mu = 3(\rho - 1) = 13 - (t_2 + 2t_3)$$

it follows that $t_2 = 1$ and $\rho - 1 = 4$.

$\mu = 2$.

Then $2 = \mathcal{Z}^* = y_1 = t_2$, and $\rho - 2 = 5$ and the type becomes $[8 * 11, 1; 3^{13}, 2^2]$.

6.2.3 case when $g = 2$

Assume that $g = 2$. Then

$$\mathcal{Z}^* = 3 - \mu.$$

The type becomes $[8 * 11, 1; 3^{13}, 2]$.

6.2.4 case when $g = 3$

Assume that $g = 3$. Then

$$\mathcal{Z}^* = (1 - \mu)\overline{g} + 2 = 4 - 2\mu.$$

The type becomes $[8 * 11, 1; 3^{13}]$.

6.3 case when $i = 1$

Suppose that $k = A - 1$. Then

$$\mathcal{Z}^* = 1 + \mu + (1 - \mu)\overline{g} + 2.$$

Then

$$\overline{g} \leq \frac{1 + \mu}{\mu - 1} = 1 + \frac{2}{\mu - 1} \leq 3.$$

6.3.1 case when $g = 0$

Suppose that $g = 0$. Then

$$\mathcal{Z}^* = 2\mu + 2.$$

Define $r' = \sum_{j=2}^{\nu_1 - 1} t_j$, where $r = r' + t_{\nu_1}$. Then $r' > 0$ by hypothesis.

Suppose that $r' = 1$. Then $\mathcal{Z}^* = j(\mu - j)$ for some $j < \mu = \nu_1 - 1$ where $\mu - j \geq j$. Hence,

$$\mu = j + 2 + \frac{6}{j - 2}.$$

But

$$\begin{aligned} \overline{Y} = 8 \cdot \mu + 2k + 2 &= (j - 1)t_j + (\mu - j + 1)t_{\mu - j + 2} + (r - 1) \cdot \mu \\ &= (j - 1) + (\mu - 2j + 2)t_{\mu - j + 2} + (r - 1) \cdot \mu. \end{aligned}$$

Therefore,

$$(\rho - 1) \cdot \mu = 2k + 2 - (j - 1) - (\mu - 2j + 2)t_{\mu - j + 2} = 15 - j - (\mu - 2j + 2)t_{\mu - j + 2}.$$

Following the next table, we obtain a solution $j = 5, \mu = 9, t_6 = 1$.

Table 2:

$j-2$	j	$j+2$	$6/(j-2)$	μ	$2k+3-j$	$2k+1+j-\mu$
1	3	5	6	11	12	5
2	4	6	3	9	11	8
3	5	7	2	9	10	9

Thus, $\nu_1 = 10, \sigma = 22, e = 32$. The type becomes $[22 * 32, 1; 10^9, 6]$.

Assume $r' \geq 2$. Then $\mathcal{Z}^* = \mu - 1 + j(\mu - j)$ for some $j < \mu = \nu - 1$ where $\mu + 2 - j \geq j$. Thus

$$2\mu + 2 \geq \mu - 1 + j(\mu - j).$$

Hence,

$$j^2 + 3 \geq (j-1)\mu \geq 2(j-1)j.$$

Therefore, $2j + 3 \geq j^2$, which implies $j \leq 3$. Moreover,

$$\mu \leq \frac{j^2+3}{j-1} = j + 1 + \frac{4}{j-1}.$$

Hence, $\mu \leq 7$. Then $y_j = 0$ for $j > 3$.

Let $a = y_1, b = y_2, c = y_3$. Thus

$$\mathcal{Z}^* = 2\mu + 2 = a(\mu - 1) + 2b(\mu - 2) + 3c(\mu - 3).$$

If $c > 0$ then $\mu \geq 6$ and

$$(a + 2b + 3c - 2)\mu = a + 4b + 9c + 2 \geq 6(a + 2b + 3c - 2).$$

Hence, $14 \geq 5a + 8b + 9c$, which induces $c = 1, b = 0, a = 1$. But this contradicts $r' > 1$. Accordingly, $c = 0$.

If $b > 0$ then $\mu \geq 4$.

By $(a + 2b - 2)\mu = a + 4b + 2 \geq 4(a + 2b - 2)$, we obtain $10 \geq 3a + 4b$. Hence, $b = 1, 2$.

If $b = 2$ then $a = 0$ and thus $\mu = 5$. By the way , from

$$\overline{Y} = 8 \cdot \mu + 2k + 2 = 2y2(\mu - 2) + (r - 2)\mu, y_2 = t_3 + t_4 = 2$$

it follows that

$$5(\rho - 2) = (\rho - 2)\mu = 2k + 2 \cdot 2 - 2t_3 - 3t_4 = 10 - t_4.$$

Therefore, $t_4 = 0, \rho - 2 = 2$. Hence, the type becomes $[14 * 20, 1; 6^{10}, 3^2]$.

If $b = 1$ then $a = 1, 2$ and thus $\mu = 1 + \frac{a}{6}$. Moreover, from $\overline{Y} = t_2 + (\mu - 1)t_\mu + (\mu - 2)t_{\mu-1} + (r - 1 - a)\mu$ it follows that

$$(\rho - a - 1)\mu = 2k + 2 - a - 2 - (\mu - 2)t_\mu - (\mu - 4)t_{\mu-1}.$$

If $a = 1$ then $\mu = 7$ and we derive a contradiction.
If $a = 2$ then $\mu = 4$ and the type becomes $[12 * 17, 1; 5^{10}, 4, 3, 2]$.

If $b = c = 0$ then $2\mu + 2 = a(\mu - 1)$. Hence, $\mu = 1 + \frac{4}{a-2}$.
Moreover, from $\overline{Y} = t_2 + (\mu - 1)t_\mu + (r - a)\mu$, it follows that

$$(\rho - a)\mu = 14 - a - (\mu - 2)t_\mu.$$

Table 3:

$a-2$	a	$4/(a-2)$	μ	$14-a$	$14-a-(\mu-2)$	$14-a-2(\mu-2)$
1	3	4	5	11	8	5
2	4	2	3	10	9	8
4	6	1	2	8	8	8

From this table, the condition that $14 - a - (\mu - 2)t_\mu$ is a multiple of μ is satisfied we we get 1) $a = 3, \mu = 5, t_5 = 2$, 2) $a = 4, \mu = 3, t_3 = 1$ or 3) $a = 6, \mu = 2, t_2 = 6$.

1) The type becomes $[14 * 20, 1; 6^9, 4, 5^2, 2]$.
2) The type becomes $[10 * 14, 1; 4^{11}, 4, 3, 2^3]$.
3) The type becomes $[8 * 11, 1; 3^{12}, 2^6]$.

6.3.2 case when $g = 1$

Suppose that $g = 1$. Then

$$\mathcal{Z}^* = \mu + 3.$$

By $\mathcal{Z}^* = \mu + 3 \geq 2\mu - 4$, we obtain $\mu \leq 7$.

$\mu = 7$
Then $\mathcal{Z}^* = 10$ and $\mathcal{Z}^* = 6y_1 + 10y_2 + 12y_3$. Hence, $y_2 = t_3 + t_6 = 1$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 2t_3 + 5t_6 + (r-1) \cdot \mu = 2 + 3t_6 + (r-1) \cdot \mu.$$

Therefore,

$$7(\rho - 1) = (\rho - 1) \cdot \mu = 2k - 1 - 3t_6 = 11 - 3t_6.$$

Here are no solutions.

$\mu = 6$
Then $\mathcal{Z}^* = 9$ and $\mathcal{Z}^* = 5y_1 + 8y_2 + 9y_3$. Hence, $y_3 = t_4 = 1$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 3t_4 + (r-1) \cdot \mu = 3 + (r-1) \cdot \mu.$$

Therefore,

$$6(\rho - 1) = (\rho - 1) \cdot \mu = 2k - 2 = 10.$$

Here are no solutions.

$\mu = 5$
Then $\nu_1 = 6$,$\mathcal{Z}^* = 8$ and $\mathcal{Z}^* = 4y_1 + 6y_2$. Hence, $y_1 = t_2 + t_5 = 2$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = t_2 + 4t_5 + (r-1) \cdot \mu = 2 + 3t_5 + (r-2) \cdot \mu.$$

Therefore,

$$5(\rho - 2) = (\rho - 2) \cdot \mu = 11 - 3t_5.$$

Then $t_5 = 2$ and $\rho - 2 = 1$. Hence, $r = 11$, $\sigma = 12 + 2 = 14, e = 14 + 6$. Hence, the type becomes $[14 * 20, 1; 6^9, 5^2]$.

$\mu = 4$
Then $\nu_1 = 5$,$\mathcal{Z}^* = 7$ and $\mathcal{Z}^* = 3y_1 + 4y_2$. Hence, $y_1 = t_2 + t_4 = 1, y_2 = t_3 = 1$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = t_2 + 3t_4 + 2 + (r-2) \cdot \mu = 3 + 2t_4 + (r-2) \cdot \mu.$$

Therefore,

$$4(\rho - 2) = (\rho - 2) \cdot \mu = 10 - 2t_4.$$

Then $t_4 = 1$ and $\rho - 2 = 2$. Hence, $r = 12$, $\sigma = 10 + 2 = 12, e = 12 + 5 = 17$. Hence, the type becomes $[12 * 17, 1; 5^{10}, 4, 3]$.

$\mu = 3$
Then $\nu_1 = 4, \mathcal{Z}^* = 6$ and $\mathcal{Z}^* = 2y_1$. Hence, $y_1 = t_2 + t_3 = 3$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = t_2 + 2t_3 + (r-3) \cdot \mu = 3 + t_3 + (r-3) \cdot \mu.$$

Therefore,

$$3(\rho - 3) = (\rho - 3) \cdot \mu = 10 - t_3.$$

Then $t_3 = 1$ and $\rho - 3 = 3$. Hence, $r = 14$, $\sigma = 8 + 2 = 10, e = 10 + 4 = 14$. Hence, the type becomes $[10 * 14, 1; 4^{10}, 3, 2^2]$.

$\mu = 2$
Then $\nu_1 = 3, \mathcal{Z}^* = 5$ and $\mathcal{Z}^* = y_1$. Hence, $y_1 = t_2 = 5$.

$$\overline{Y} = 8 \cdot \mu + 2k + 1 = 5 + (r-5) \cdot \mu.$$

Therefore,

$$2(\rho - 5) = (\rho - 5) \cdot \mu = 8.$$

Then $\rho - 5 = 4$. Hence, $r = 17$, $\sigma = 6 + 2 = 8, e = 11$. Hence, the type becomes $[8 * 11, 1; 3^{13}, 2^5]$.

6.3.3 case when $g = 2$

Suppose that $g = 2$. Then $\mathcal{Z}^* = 4 \geq \mu - 1$. Hence, $5 \geq \mu$.

$\mu = 5$
Then $\nu_1 = 6$ and $\mathcal{Z}^* = 4y_1 + 6y_2$. Hence, $y_1 = t_2 + t_5 = 1$.

$$\overline{Y} = 8 \cdot \mu + 2k = t_2 + 4t_5 + (r-1) \cdot \mu = 1 + 3t_5 + (r-1) \cdot \mu.$$

Therefore,

$$5(\rho - 1) = (\rho - 1) \cdot \mu = 11 - 3t_5.$$

Here are no solutions.

$\mu = 4$
Then $\nu_1 = 5$ and $4 = \mathcal{Z}^* = 3y_1 + 4y_2$. Hence, $y_2 = t_3 = 1$.

$$\overline{Y} = 8 \cdot \mu + 2k = t_2 + 2t_3 + (r-1) \cdot \mu = 2 + (r-1) \cdot \mu.$$

Therefore,

$$4(\rho - 1) = (\rho - 1) \cdot \mu = 10.$$

Here are no solutions.

$\mu = 3$
Then $\nu_1 = 4$ and $4 = \mathcal{Z}^* = 2y_1$. Hence, $y_1 = t_2 + t_3 = 2$.

$$\overline{Y} = 8 \cdot \mu + 2k = t_2 + 2t_3 + (r-2) \cdot \mu = 2 + t_3 + (r-2) \cdot \mu.$$

Therefore,
$$3(\rho - 1) = (\rho - 1) \cdot \mu = 10 - t_3.$$
Then $t_3 = 1, \rho - 1 = 3$; $\rho = 5$. Hence, the type becomes $[10 * 14, 1; 4^{11}, 3, 2]$.

$\mu = 2$
Then $\nu_1 = 3$ and $4 = \mathcal{Z}^* = y_1$. Hence, $y_1 = t_2 = 4$.

$$\overline{Y} = 8 \cdot \mu + 2k = 4 + (r-4) \cdot \mu.$$

Therefore,
$$2(\rho - 4) = (\rho - 4) \cdot \mu = 12 - 4 = 8.$$
Then $\rho - 4 = 4$; $\rho = 8$. Hence, the type becomes $[8 * 11, 1; 3^{12}, 2^4]$.

6.3.4 case when $g = 3$

Suppose that $g = 3$. If $\mathcal{Z}^* \neq 0$, then $\mathcal{Z}^* = 5 - \mu \geq \mu - 1$. Hence, $3 \geq \mu$.
Otherwise, $\mathcal{Z}^* = 0$; $\mu = 5$.
If $\mu = 3$ then the type becomes $[10 * 14, 1; 4^{11}, 3]$.

6.3.5 case when $g = 4$

Suppose that $g = 4$. Then $\mathcal{Z}^* = 6 - 2\mu \geq \mu - 1$. Hence, $3 \geq \mu$.
If $\mu = 2$ then the type becomes $[8 * 11, 1; 3^{12}, 2^2]$.
If $\mu = 3$ then $3\mu = 2k - 2 = 10$. A contradiction.

7 Class IIIb.

Class IIIb. We say that the type belongs to Class IIIb if $p \geq 2$ except for the condition of Class III_a.
Then $i = A - k \geq 0$ and $\tilde{q} = \tilde{k} - k \geq 0$. Thus,

1. $\overline{Y} = 8\mu + k + A_1 = 8\mu + 2k + i - \overline{g}$,

2. $\overline{X} = 8\mu^2 + 2k\mu + \tilde{q} - i - \overline{g}$.

Therefore,

$$\mathcal{Z}^* = i(\mu+1) + (1-\mu)\overline{g} - \tilde{q}.$$

Putting $\tilde{i} = i - \overline{g}$, we get $i + \overline{g} = \tilde{i} + 2\overline{g}$ and so $\overline{Y} = 8\mu + 2k + \tilde{i}$. Hence,

$$\mathcal{Z}^* = \tilde{i}(\mu+1) + 2\overline{g} - \tilde{q}.$$

Since $\tilde{q} \geq 0$, it follows that

$$\tilde{i}(\mu+1) + 2\overline{g} = \mathcal{Z}^* + \tilde{q} \geq 0.$$

Therefore,

$$(i - \overline{g})\nu_1 = \tilde{i}(\mu+1) \geq -2\overline{g}.$$

Thus,

$$(\mu+1)i \geq (\mu-2)\overline{g}.$$

Suppose that $\overline{g} \geq 0$. Then

$$\frac{i(\mu+1)}{\mu-1} \geq \overline{g}.$$

If $i = 0$ then $g = 1$.
If $i = 1$ then $g \leq 4$.

7.1 case when $\overline{g} = 2$

Suppose that $g = 3$;thus $\overline{g} = 2$ and $\tilde{i} = i - \overline{g} = 1 - 2 = -1$. By $\frac{\nu_1}{\nu_1 - 2} = \frac{\mu+1}{\mu-1} \geq \overline{g} = 2$, we get $\mu \leq 3$.

$\mu = 3$
Then $\nu_1 = 4$ and

$$\mathcal{Z}^* = \tilde{i}(\mu+1) + 2\overline{g} - \tilde{q} = -4 + 4 - \tilde{q} = -\tilde{q}.$$

Therefore, $\tilde{q} = 0$ and $\mathcal{Z}^* = 0$.

Hence, $3\rho = \rho\mu = 2k + \tilde{i} = 2k - 1$. But from $\tilde{q} = 0$, it follows that 1) $k = 8, p = 2$ or 2) $k = 9, p = 3$. Thus,$k = 8, p = 2$ and $3\rho = 15$, which implies that $\rho = 5, r = 13$.

If $B = 1$ then $u = 1, \sigma = 8 + 2 = 10$ and $e = 10 + 4 + 1 = 15$. Therefore, the type becomes $[10 * 15, 1; 4^{13}]$.

If $B = 0$ then $u = 0, \sigma = 8 + 2 = 10$ and $e = 10$. Therefore, the type becomes $[10 * 10; 4^{13}]$.

$\mu = 2$

Then $\nu_1 = 3$ and

$$\mathcal{Z}^* = \tilde{\tilde{i}}(\mu + 1) + 2\overline{g} - \tilde{q} = -3 + 4 - \tilde{q} = 1 - \tilde{q}.$$

Therefore, either A) $\tilde{q} = 0$ and $\mathcal{Z}^* = 1$ or B) $\tilde{q} = 1$ and $\mathcal{Z}^* = 0$.

A) $\tilde{q} = 0$ and $\mathcal{Z}^* = 1$.

If $\tilde{q} = 0$ then $\mathcal{Z}^* = 1$. Hence, $t_2 = 1$ and $\overline{Y} = (r-1)\mu + 1 = 8\mu + 2k + 1 - \overline{g} = 8\mu + 2k - 1$. Therefore, $2(\rho - 1) = 2k - 2$ and $\rho = k$. But from $\tilde{q} = 0$, it follows that 1) $k = 8, p = 2$ or 2) $k = 9, p = 3$.

1) If $B = 1$ then $u = 1, \rho = k = 8$ and $\sigma = 6+2 = 8$ and $e = 8+3+1 = 12$.

Therefore, $r = 16$ and the type becomes $[8 * 12, 1; 3^{15}, 2]$.

If $B = 0$ then $u = 0, \rho = k = 8$ and $\sigma = 6 + 2 = 8$ and $e = 8$.

Therefore, $r = 16$ and the type becomes $[8 * 8; 3^{15}, 2]$.

2) $k = 9, p = 3, B = 1, u = 0$. Then $\rho = k = 9$ and $\sigma = 6 + 3 = 9$ and $e = 9 + 3 = 12$. Therefore, $r = 17$ and the type becomes $[9 * 12, 1; 3^{16}, 2]$.

B) $\tilde{q} = 1$

From $\tilde{q} = 1$, it follows that $k(p-1) - 2p^2 = 1$. Thus,

$$k = 2(p+1) + \frac{3}{p-1}.$$

Thus $p - 1 = 1$ or 3.

If $p = 2$ then $k = 9$. But $k - wp = 9 - 2w = 2u$, a contradiction.

If $p = 4$ then $k = 11$. But $k - wp = 11 - 4w = 2u$, a contradiction.

7.2 case when $\overline{g} = 1$

Suppose that $g = 2$;thus $\overline{g} = 1$ and $\tilde{\tilde{i}} = i - \overline{g} = 1 - 1 = 0$. Then

$$\mathcal{Z}^* = 2 - \tilde{q}.$$

Therefore $\tilde{q} = 2$ or 1 or 0.

7.2.1 case when $\tilde{q} = 2$

If $\tilde{q} = 2$ then $\mathcal{Z}^* = 0$. From $\tilde{q} = 2$, it follows that $k(p-1) - 2p^2 = 2$. Thus,

$$k = 2(p+1) + \frac{4}{p-1}.$$

Table 4:

p	$2(p+1)$	$4/(p-1)$	k	$(k-3p)/2$	$(k-4p)/2$
2	6	4	10	2	1
3	8	2	10	0.5	-1
5	12	1	13	-1	-3.5

Thus we get the next table.

Thus, $p = 2, k = 10$. If $B = 1$ then $u = 2$ and if $B = 0$ then $u = 1$. From

$$\overline{Y} = 8\mu + 2k + i - \overline{g} = 8\mu + 2k = r\mu.$$

it follows that $\rho\mu = 2k = 20$.

Table 5: $B = 1, u = 2$

ρ	μ	ν_1	σ	e	r
1	20	21	44	67	9
2	10	11	24	37	10
4	5	6	14	22	12
5	4	5	12	19	13
10	2	3	8	13	18

Table 6: $B = 0, u = 1$

ρ	μ	ν_1	σ	e	r
1	20	21	44	45	9
2	10	11	24	25	10
4	5	6	14	15	12
5	4	5	12	13	13
10	2	3	8	9	18

7.3 case when $\overline{g} = 3$

Suppose that $g = 4$;thus $\overline{g} = 3$ and $\tilde{i} = i - \overline{g} = 1 - 3 = -2$. Then since $\frac{\nu_1 i}{\nu_1 - 2} \geq \overline{g} = 3$, it follows that $\nu_1 = 3$. Hence,

$$\mathcal{Z}^* = \tilde{i}(\mu + 1) + 2\overline{g} - \tilde{q} = -2 \cdot 3 + 6 - \tilde{q} = -\tilde{q}.$$

Therefore, $\tilde{q} = 0$ and $\mathcal{Z}^* = 0$. Hence, $\rho\mu = 2\rho = 2k - 2$; so $\rho = k - 1$.

By $\tilde{q} = \tilde{k} - k = 0$, we get either 1) $p = 2, k = 8$ or 2) $p = 3, k = 9$.

If $p = 2, k = 8$ then $\rho = k - 1 = 7$ and $\nu_1 = 3$.

If $B = 1$ then $\sigma = 6 + 2 = 8, u = 1$ and $e = 8 + 3 + 1$. Hence the type becomes $[8 * 12, 1; 3^{15}]$.

If $B = 0$ then $\sigma = 6 + 2 = 8, u = 0$ and $e = 8$. Hence the type becomes $[8 * 8; 3^{15}]$.

If $p = 3, k = 9$ then $\rho = k - 1 = 8$ and $\nu_1 = 3$ and moreover,$\sigma = 6 + 3 = 9, u = 1$ and $e = 9 + 3 =$. Hence the type becomes $[9 * 12, 1; 3^{16}]$.

7.3.1 case when $\tilde{q} = 1$

As was shown before, this case does not occur.

7.3.2 case when $\tilde{q} = 0$

If $\tilde{q} = 0$ then 1) $k = 8, p = 2$ or 2)$k = 9, p = 3$. Moreover, $\mathcal{Z}^* = 2 - \tilde{q} = 2$. Since $\mathcal{Z}^* \geq \mu - 1$, it follows that $\mu = 2$ or 3.

$\mu = 2$.

Then $\nu_1 = 3$ and $\mathcal{Z}^* = t_2$. Hence, $t_2 = 2$ and

$$\overline{Y} = 8\mu + 2k = 2 + (r - 2)\mu.$$

Hence, $(\rho - 2)\mu = 2(\rho - 2) = 2k - 2$. Thus, $\rho = k + 1$.

1) $k = 8, p = 2$. Then $\rho = k + 1 = 9; r = 17$.

If $B = 1$ then $\sigma = 6 + 2$, $e = 8 + 3 + 1 = 12$ and the type becomes $[8 * 12, 1; 3^{15}, 2^2]$.

2) $k = 9, p = 3$. Then $\rho = k + 1 = 10; r = 18$ and $\sigma = 6 + 3 = 9$, $e = 9 + 3 = 12$ and the type becomes $[9 * 12, 1; 3^{16}, 2^2]$.

$\mu = 3$.

Then $\nu_1 = 4$ and $\mathcal{Z}^* = 2(t_2 + t_3)$. Hence, $t_2 + t_3 = 1$. From

$$\overline{Y} = 8\mu + 2k = t_2 + 2t_3 + (r-1)\mu.$$

Hence, $(\rho - 1)\mu = 3(\rho - 1) = 2k - (t_2 + 2t_3) = 2k - 1 - t_3$. Thus, $3(\rho - 1) = 2k - 1 - t_3$.

1) $k = 8, p = 2$. Then $3(\rho - 1) = 2k - 1 - t_3 = 15 - t_3$. Hence, $t_3 = 0$ and $t_2 = 1, \rho - 1 = 5; r = 14$

If $B = 1$ then $\sigma = 8+2 = 10$, $e = 10+4+1 = 15$ and the type becomes $[10 * 15, 1; 4^{13}, 2]$.

If $B = 0$ then $\sigma = 8+2 = 10$, $e = 10$ and the type becomes $[10*10; 4^{13}, 2]$.

2) $k = 9, p = 3, B = 1, u = 0$. Then $3(\rho - 1) = 2k - 1 - t_3 = 17 - t_3$. No solution.

7.4 case when $\overline{g} = 0$

Suppose that $g = 1$;thus $\overline{g} = 0$ and $\tilde{i} = i - \overline{g} = i$.

Assume $i = 0$. Then $\tilde{i} = 0$ and $\mathcal{Z}^* = -\tilde{q}$.

By $\tilde{q} = 0$, we have two cases, i.e. 1). $k = 8, p = 2, B = 1, 0$ and 2). $k = 9, p = 3, B = 1$.

$$\overline{Y} = 8\mu + 2k = r\mu.$$

Hence, $\rho\mu = 2k$.

Table 7: $k = 8, p = 2, B = 1$

ρ	r	μ	ν_1	σ	u	e
1	9	16	17	36	1	54
2	10	8	9	20	1	30
4	12	4	5	12	1	18
8	16	2	3	8	1	12

7.4.1 case when $i = 1$

Assume $i = 1$. Then

$$\mathcal{Z}^* = \mu + 1 - \tilde{q}.$$

If $\mathcal{Z}^* \neq 0$, then $\mathcal{Z}^* \geq \mu - 1$ and hence, $\tilde{q} \leq 2$.

Table 8: $k = 8, p = 2, B = 0$

ρ	r	μ	ν_1	σ	u	e
1	9	16	17	36	0	36
2	10	8	9	20	0	20
4	12	4	5	12	0	12
8	16	2	3	8	0	8

Table 9: $k = 9, p = 3, B = 1$

ρ	r	μ	ν_1	σ	u	e
1	9	18	19	41	0	60
2	10	9	10	23	0	33
3	11	6	7	17	0	24
6	14	3	4	11	0	15
9	17	2	3	9	0	12

7.4.2 case when $\tilde{q} = 2$

Suppose that $\tilde{q} = 2$. Then $k = 10, p = 2$ and $\mathcal{Z}^* = \mu - 1$.

Moreover,from $\mathcal{Z}^* = y_1(\mu - 1)$, it follows that $y_1 = t_2 + t_\mu = 1$ and

$$\overline{Y} = 8\mu + 2k + 1 = t_2 + (\mu - 1)t_\mu + (r - 1)\mu = 1 + (\mu - 2)t_\mu + (r - 1)\mu.$$

Therefore, $(\rho - 1)\mu = 2k - (\mu - 2)t_\mu$. Since $k = 10$ and $t_\mu = 0$ or 1, it follows that 1) $(\rho - 1)\mu = 2k = 20, t_2 = 1$ or 2) $\rho\mu = 2k + 2 = 22, t_\mu = 1$.

1). From $(\rho - 1)\mu = 2k = 20, t_2 = 1$, we obtain the following tables:

Then the types are as follows:

1. $[44 * 67, 1; 21^9, 2]$,
2. $[24 * 37, 1; 11^{10}, 2]$,
3. $[14 * 22, 1; 6^{12}, 2]$,
4. $[12 * 19, 1; 5^{13}, 2]$,
5. $[8 * 13, 1; 3^{18}, 2]$.

Table 10: $B = 1, u = 2, p = 2$

ρ	μ	ν_1	σ	e	r
2	20	21	44	67	10
3	10	11	24	37	11
5	5	6	14	22	13
6	4	5	12	19	14
11	2	3	8	13	19

Table 11: $B = 0, u = 1, p = 2$

ρ	μ	ν_1	σ	e	r
2	20	21	44	45	10
3	10	11	24	25	11
5	5	6	14	15	13
6	4	5	12	13	14
11	2	3	8	9	19

Then the types are as follows:

1. $[44 * 45; 21^9, 2]$,
2. $[24 * 25; 11^{10}, 2]$,
3. $[14 * 15; 6^{12}, 2]$,
4. $[12 * 13; 5^{13}, 2]$,
5. $[8 * 9; 3^{18}, 2]$.

2). From $\rho\mu = 2k + 2 = 22, t_\mu = 1$, we obtain the following tables:
Then the types are as follows:

1. $[48 * 73, 1; 23^8, 22]$,
2. $[26 * 40, 1; 11^9, 10]$,
3. $[8 * 13, 1; 3^{18}, 2]$,

Table 12: $B = 1, u = 2, p = 2$

ρ	μ	ν_1	σ	e	r
1	22	23	48	73	9
2	11	12	26	40	10
11	2	3	8	13	19

Table 13: $B = 0, u = 1, p = 2$

ρ	μ	ν_1	σ	e	r
1	22	23	48	49	9
2	11	12	26	27	10
11	2	3	8	9	19

Then the types are as follows:

1. $[48 * 49; 23^8, 22]$,
2. $[26 * 27; 11^9, 10]$,
3. $[8 * 9; 3^{18}, 2]$,

7.4.3 case when $\tilde{q} = 1$

As was proved before,the case when $\tilde{q} = 1$ never occur.

7.4.4 case when $\tilde{q} = 0$

Suppose that $\tilde{q} = 0$. Then either 1) $k = 8, p = 2$ or 2) $k = 9, p = 3$ and $\mathcal{Z}^* = \mu + 1$.

If $\mu \geq 4$ then $\mu \geq 4\mu + 1 \geq 2\mu - 4$. Thus $\mu \leq 5$.

$\mu = 5$.

Then $\mathcal{Z}^* = \mu + 1 = 6$ and $\mathcal{Z}^* = 4y_1 + 6y_2$. Hence, $y_1 = 0, y_2 = t_3 + t_4 = 1$. Therefore, from

$$\overline{Y} = 8\mu + 2k + 1 = 2t_3 + 3t_4 + (r - 1)\mu = 2 + t_4 + (r - 1)\mu$$

it follows that $5(\rho - 1) = (\rho - 1)\mu = 2k - 1 - t_4$.

1) $k = 8, p = 2$.

Then $5(\rho - 1) = 2k - 1 - t_4 = 15 - t_4$. Hence, $t_4 = 0, t_3 = 1$. $\rho - 1 = 3$ and $\sigma = 12 + 2$.

If $B = 1$ then $e = 21$. The type becomes $[14 * 21, 1; 6^{11}, 3]$,

If $B = 0$ then $e = 15$. The type becomes $[14 * 15; 6^{11}, 3]$,

2) $k = 9, p = 3, B = 1$.

Then $5(\rho - 1) = 2k - 1 - t_4 = 17 - t_4$. No solution.

$\mu = 4$.

Then $\mathcal{Z}^* = \mu + 1 = 5$ and $\mathcal{Z}^* = 3y_1 + 4y_2$. No solution.

$\mu = 3$.

Then $\nu_1 = 4$ and $\mathcal{Z}^* = \mu + 1 = 4$ and $\mathcal{Z}^* = 2y_1 = 4$. Then $y_1 = t_2 + t_3 = 2$. By

$$\overline{Y} = 8\mu + 2k + 1 = t_2 + 2t_3 + (r - 2)\mu = 2 + t_3 + (r - 2)\mu$$

we obtain

$$3(\rho - 2) = 2k - 1 - t_3.$$

1) $k = 8$. Then $t_3 = 0, t_2 = 2$ and $\rho = 7$. Therefore, $\sigma = 10, e = 10 + 4 + 1 = 15, r = 15$ for $B = 1$. When $B = 1$,the type turns out to be $[10*15, 1; 4^{13}, 2^2]$. But if $B = 0$, the type turns out to be $[10 * 10;^{13}, 2^2]$.

2) $k = 9$. Then $3(\rho - 2) = 2k - 1 - t_3 = 17 - t_3$. Hence, $t_3 = 2, t_2 = 0$ and $\rho = 7$. $\sigma = 11, e = 11 + 4 = 15, r = 15$ for $B = 1, u = 0$.

7.4.5 case when $\mathcal{Z}^* = 0$

Suppose that $\mathcal{Z}^* = 0$. Then $0 = \mathcal{Z}^* = \mu + 1 - \tilde{q}$. Hence, $\mu + 1 = \tilde{q} = k(p - 1) - 2p^2$. Thus

$$\mu + 1 = k(p - 1) - 2p^2 \tag{21}$$

By $\overline{Y} = 8\mu + 2k + 1 = r\mu$, we obtain

$$\rho\mu = 2k + 1. \tag{22}$$

Hence,

$$\rho\mu + \rho = k(\rho p - \rho) - 2\rho p^2.$$

Therefore,

$$2k + 1 + \rho = k(\rho p - \rho) - 2\rho p^2.$$

Hence,

$$1 + \rho + 2\rho p^2 = k(\rho p - \rho - 2). \tag{23}$$

This implies that ρ is odd.

7.4.6 case when $\rho = 1$

Suppose that $\rho = 1$.Then $2 + 2p^2 = k(p - 3)$. Hence, $p \geq 4$ and

$$k = 2(p + 3) + \frac{20}{p - 3}.$$

Table 14: $B = 1$

p	k	u	μ	ν_1	σ	e
4	34	11	69	70	144	225
7	25	2	51	52	111	165
8	26	1	53	54	116	171

Then the types are as follows:

1. $[144 * 225, 1; 70^9]$,
2. $[111 * 165, 1; 52^9]$,
3. $[116 * 171, 1; 54^9]$,

Table 15: $B = 0$

p	k	μ	ν_1	σ	e
4	34	69	70	144	162
5	26	53	54	113	119

Then the types are as follows:

1. $[144 * 162; 70^9]$,
2. $[113 * 119; 54^9]$.

7.4.7 case when $\rho = 3$

Suppose that $\rho = 3$. Then

$$6p^2 + 4 = k(3p - 5).$$

Since $6p^2+4=(3p-5)(2p+3)+p+19$, it follows that $k=2p+3+m, m=\frac{p+19}{3p-5}$.

If $p=2$ then $m=21$ and $k=28$. From $3\mu=\rho\mu=2k+1=57$, it follows that $\mu=19$; thus $\nu_1=20, \sigma=40+2=42$.

If $B=1$ then $u=11$ and $e=42+20+11=73$; the type becomes $[42*73, 1; 20^{11}]$.

If $B=0$ then $u=10$ and $e=42+10=52$; the type becomes $[42*52; 20^{11}]$.

If $m=2$ then $p=\frac{19}{5}$, a contradiction. Thus $m\geq 3$. Hence, $p+19\geq 3(3p-5)$. So, $p\leq 4$. But $p=3$ and $p=4$ cannot be solutions.

7.4.8 case when $\rho\geq 5$

Suppose that $\rho\geq 5$. From $k\geq 3p$, it follows that

$$1+\rho+2\rho p^2=k(\rho p-\rho-2)\geq 3p(\rho p-\rho-2)=3\rho dp^2-3p(\rho+2).$$

By $\rho\geq 5$, we get $p\leq 3$.

If $p=2$ then $1+9\rho=(\rho-2)k$; hence, $(\rho-2)(k-9)=19$. Therefore, $k-9=1, \rho-2=19$. Then $\rho=21$ and $21\mu=\rho\mu=2k+1=21$; so $\mu=1$.

If $p=3$ then $1+19\rho=2(\rho-1)k$; hence, $(\rho-1)(2k-19)=20$. Therefore, 1) $\rho-1=4, 2k-19=5$ or 2) $\rho-1=20, 2k-19=1$.

1) $\rho=5$ and $k=12$.

By $5\mu=\rho\mu=2k+1=25$; so $\mu=5$. Hence, $B=0$ and $u=0$. Moreover, $\sigma=12+3=15, e=15, r=5+8=13$.

Then the type is $[15*15; 6^{13}]$.

2) $\rho=21, k=10$.

By $21\mu=\rho\mu=2k+1=21$, we get $\mu=1$.

If $p=4$ then $1+33\rho=(3\rho-2)k$; hence, $(3\rho-2)(k-11)=23$. Therefore, $3\rho-2=23$ and $k-11=1$. A contradiction.

7.5 case when $\overline{g}=-1, i=0$

Suppose that $g=0$ and $i=0$; thus $\tilde{i}=\overline{g}=1$. In this case,

$$\mathcal{Z}^*=\mu-1-\tilde{q}.$$

I). If $\mathcal{Z}^* \neq 0$, then $\mathcal{Z}^* \geq \mu - 1$ and hence, $\tilde{q} = 0$ and $\mathcal{Z}^* = \mu - 1$. Thus either 1) $k = 8, p = 2$ or 2) $k = 9, p = 3$. Therefore, by $\mathcal{Z}^* = y_1(\mu - 1)$ we obtain $y_1 = t_2 + t_\mu = 1$.

By $\overline{Y} = 8\mu + 2k + 1 = t_2 + (\mu - 1)t_\mu + (r - 1)\mu$, we obtain

$$(\rho - 1)\mu = 2k + 1 - t_2 + (\mu - 1)t_\mu = 2k + (\mu - 2)t_\mu.$$

Suppose that $t_\mu = 0$; hence,$t_2 = 1$. Hence,

$$(\rho - 1)\mu = 2k.$$

1) $k = 8, p = 2$.
$B = 1$.

Table 16: $k = 8, p = 2, B = 1$

$\rho - 1$	ρ	μ	ν_1	σ	e
1	2	16	17	36	54
2	3	8	9	20	30
4	5	4	5	12	18
8	9	2	3	8	12

Table 17: $k = 8, p = 2, B = 0$

$\rho - 1$	ρ	μ	ν_1	σ	e
1	2	16	17	36	36
2	3	8	9	20	20
4	5	4	5	12	12
8	9	2	3	8	8

2) $k = 9, p = 3$.

Suppose that $t_\mu = 1$; hence,$t_2 = 0$ and

$$\rho\mu = 2k + 2.$$

1) $k = 8, p = 2$.
2) $k = 9, p = 3$.

Table 18: $k = 9, p = 3, B = 1$

$\rho - 1$	ρ	μ	ν_1	σ	e
1	2	18	19	41	60
2	3	9	10	23	33
3	4	6	7	17	24
6	7	3	4	11	15
9	10	2	3	9	12

Table 19: $k = 8, p = 2, B = 1$

ρ	μ	ν_1	σ	e	$\tilde{B}$
1	18	19	40	60	80
2	9	10	22	33	44
3	6	7	16	24	32
6	3	4	10	15	20
9	2	3	8	12	16

Table 20: $k = 8, p = 2, B = 0$

ρ	μ	ν_1	σ	e
1	18	19	40	40
2	9	10	22	22
3	6	7	16	16
6	3	4	10	10
9	2	3	8	8

Table 21: $k = 9, p = 3, t_\mu = 1$.

ρ	μ	ν_1	σ	e
1	20	21	45	66
2	10	11	25	36
4	5	6	15	21
5	4	5	13	18
10	2	3	9	12

II). If $\mathcal{Z}^* = 0$, then $\tilde{q} = \mu - 1$.

By

$$\overline{Y} = 8\mu + 2k + 1 = r\mu$$

we obtain $\rho\mu = 2k + 1$.

From

$$\mu - 1 = \tilde{q} = k(p-1) - 2p^2$$

it follows that

$$\rho\mu - \rho = k\rho(p-1) - \rho 2p^2.$$

Hence,

$$2k + 1 = k\rho(p-1) - \rho 2p^2 + \rho.$$

Thus,

$$2\rho p^2 - \rho + 1 = k(\rho(p-1) - 2). \tag{24}$$

This implies that ρ is odd.

7.5.1 case when $\rho = 1$

Suppose that $\rho = 1$. Then $2p^2 = k(p-3)$. Hence,

$$k = 2(p+3) + \frac{18}{p-3}.$$

Table 22: $\rho = 1, B = 1$

$p-3$	p	$18/(p-3)$	$2p+6$	k	μ	ν_1	σ	u	e
1	4	18	14	32	65	66	136	10	212
2	5	9	16	25	51	52	109	5	166
3	6	6	18	24	49	50	106	3	159
6	9	3	24	27	55	56	121	0	177

Table 23: $\rho = 1, B = 0$

$p-3$	p	$18/(p-3)$	$2p+6$	k	μ	ν_1	σ
1	4	18	14	32	65	66	136
3	6	6	18	24	49	50	106

7.5.2 case when $\rho = 3$

Suppose that $\rho = 3$. Then $6p^2 - 2 = k(3p-5)$. Hence,

$$k = 2p + 3 + \frac{p+13}{3p-5}.$$

Denoting $\frac{p+13}{3p-5}$ by m, we obtain $k = 2p+3+m$.

If $p = 2$ then $m = 15$ and $k = 22$. By $3\mu = \rho\mu = 2k+1 = 45$, we get $\mu = 15$. Hence, $\nu_1 = 16$ and $\sigma = 32+2 = 34$.

If $B = 1$ then $u = (k-3p)/2 = 8$. Hence, $e = 34+16+8 = 58$.

The type becomes $[34 * 58, 1; 16^{11}]$ which has $g = 0, Z^2 = 21, A = 22$.

If $B = 0$ then $u = (k-4p)/2 = 7$. Hence, $e = 34+7 = 41$.

The type becomes $[34 * 41; 16^{11}]$ which has $g = 0, Z^2 = 21, A = 22$.

If $p = 3$ then $m = 4$ and $k = 13$. By $3\mu = \rho\mu = 2k+1 = 27$, we get $\mu = 9$. Hence, $\nu_1 = 10$ and $\sigma = 23$.

If $B = 1$ then $u = (k-3p)/2 = 2$. Hence, $e = 23+10+2 = 35$.

The type becomes $[23 * 35, 1; 10^{11}]$ which has $g = 0, Z^2 = 12, A = 13$.

If $B = 0$ then $u = (k-4p)/2 = 5/2$, a contradiction.

If $p \geq 4$ then there are no solutions.

7.5.3 case when $\rho \geq 5$

Suppose that $\rho \geq 5$.

From $2\rho p^2 - \rho + 1 = k(\rho(p-1)-2) \geq 3p(\rho(p-1)-2)$, it follows that

$$3p + \frac{6p}{\rho} + \frac{1}{\rho} - 1 \geq p^2.$$

By $\rho \geq 5$ we conclude $p = 2, 3$.

Suppose $p = 2$; thus $k(\rho-2) = 7\rho+1$. Hence, $(k-7)(\rho-2) = 15, \rho-2 \geq 3$.

Table 24: $\rho \geq 5, B = 1$

$\rho-2$	ρ	$k-7$	k	μ	ν_1	σ	u	e
3	5	5	12	5	6	14	3	23
5	7	3	10	3	4	10	2	16

The type becomes

1. $[14 * 23, 1; 6^{13}]$,

2. $[10 * 16, 1; 4^{15}]$.

Table 25: $\rho \geq 5, B = 0$

$\rho - 2$	ρ	$k - 7$	k	μ	ν_1	σ	u	e
3	5	5	12	5	6	14	2	16
5	7	3	10	3	4	10	1	11

The type becomes

1. $[14 * 16; 6^{13}]$,
2. $[10 * 11; 4^{15}]$.

Suppose $p = 3$; thus $2k(\rho - 1) = 17\rho + 1$. Hence, $(2k - 17)(\rho - 1) = 18$. $2k - 17$ is odd. We get the following table.

Table 26: $B = 1, p = 3$

$\rho - 1$	ρ	$2k - 17$	k	μ	ν_1	σ	u	e
2	3	9	13	9	10	23	2	35

The type becomes $[23 * 35, 1; 10^{11}]$.

7.6 case when $i = 1$

Suppose that $i = 1, g = 0$;thus $\overline{g} = -1$ and $\tilde{\overline{i}} = i - \overline{g} = i + 1 = 2$. Then

$$\mathcal{Z}^* = 2\mu - \tilde{q}.$$

If $\mathcal{Z}^* \neq 0$ and $\mu - 4 \geq 0$ then $\mathcal{Z}^* \geq 2\mu - 4$ and hence, $\tilde{q} \leq 4$.
If $\mu = 2$ then $\mathcal{Z}^* = 4 - \tilde{q} \geq 0$; thus $\tilde{q} \leq 4$.
If $\mu = 3$ then $\mathcal{Z}^* = 6 - \tilde{q} = 2y_1 \neq 0$; thus $\tilde{q} \leq 4$.

7.6.1 case when $\tilde{q} = 4$

Suppose that $\tilde{q} = 4$. Then $\mathcal{Z}^* = 2\mu - 4$. Assume $\mu \geq 4$. Since $\mathcal{Z}^* = y_1(\mu - 1) + 2y_2(\mu - 2) + 3y_3(\mu - 3) + \cdots$, it follows that $y_2 = t_3 + t_{\mu-1} = 1$.

By

$$\overline{Y} = 8\mu + 2k + 2 = 2t_3 + (\mu - 2)t_{\mu-1} + (r-1)\mu = 2 + (\mu - 4)t_{\mu-1} + (r-1)\mu,$$

we obtain $(\rho - 1)\mu = 2k + (\mu - 4)t_{\mu-1}$.

On the other hand, from $\tilde{q} = 4$, it follows that $k(p-1) - 2p^2 = 4$ and hence,

$$k = 2(p+1) + \frac{6}{p-1}.$$

Table 27:

$p-1$	p	$2(p+1)$	$6/(p-1)$	k	$(k-3p)/2$	$(k-4p)/2$
1	2	6	6	12	3	2

Thus $k = 12, p = 2$.
If $B = 0$ then $u = 2$;if $B = 0$ then $u = 3$.
Therefore, $(\rho - 1)\mu = 24 + (\mu - 4)t_{\mu-1}$.
If $t_{\mu-1} = 0$ then $(\rho - 1)\mu = 24$.

Table 28: $t_{\mu-1} = 0$

$\rho-1$	μ	ν_1	σ	e	ρ	r
1	24	25	52	80	2	10
2	12	13	28	44	3	11
3	8	9	20	32	4	12
4	6	7	16	26	5	13
6	4	5	12	20	7	15
8	3	4	10	17	9	17
12	2	3	8	14	13	21

The types are as follows

1. $[52 * 80, 1; 25^9, 3]$,
2. $[28 * 44, 1; 13^{10}, 3]$,
3. $[20 * 32, 1; 13^{11}, 3]$,
4. $[16 * 26, 1; 7^{12}, 3]$,

5. $[12 * 20, 1; 5^4, 3]$,

6. $[10 * 17, 1; 4^{16}, 3]$,

7. $[8 * 14, 1; 3^{20}, 3]$.

If $t_{\mu-1} = 1$ then $(\rho - 2)\mu = 24 + 4 = 28$.

Table 29: $t_{\mu-1} = 1, B = 1$

$\rho - 1$	μ	ν_1	σ	e	ρ	r
1	28	29	60	92	2	9
2	14	15	32	50	3	10
4	7	8	18	29	5	12
7	4	5	12	20	8	15
14	2	3	8	14	15	22

The types are as follows

1. $[60 * 92, 1; 29^9, 27]$

2. $[32 * 50, 1; 15^{10}, 13]$

3. $[18 * 29, 1; 8^{11}, 6]$

4. $[12 * 20, 1; 5^{14}, 3]$

Table 30: $t_{\mu-1} = 1, B = 0$

$\rho - 1$	μ	ν_1	σ	e	ρ	r
1	28	29	60	62	2	10
2	14	15	32	34	3	11
4	7	8	18	20	5	13
7	4	5	12	14	8	16
14	2	3	8	10	15	23
28	1	2	6	8	29	37

The types are as follows

1. $[60 * 62; 29^9, 27]$,

2. $[32 * 34; 15^{10}, 13]$,

3. $[18 * 20; 8^{11}, 6]$,

4. $[12 * 14; 5^{14}, 3]$

7.6.2 case when $\tilde{q} = 3$

Suppose that $\tilde{q} = 3$,from which it follows that $k(p-1) - 2p^2 = 3$ and hence,

$$k = 2(p+1) + \frac{5}{p-1}.$$

Hence, $p - 1 = 1$ or 5.

If $p = 2$ then $k = 11$. But $k - wp = 11 - 2w = 2u$, contradiction.

If $p = 6$ then $k = 15$. But $k - wp = 15 - 6w = 2u$, contradiction.

7.6.3 case when $\tilde{q} = 2$

Suppose that $\tilde{q} = 2$. Then $\tilde{q} = 2$, it follows that $k(p-1) - 2p^2 =$ and hence,

$$k = 2(p+1) + \frac{4}{p-1}.$$

Table 31: $\tilde{q} = 2$

$p-1$	p	$2(p+1)$	$4/(p-1)$	k	$(k-3p)/2$	$(k-4p)/2$
1	2	6	4	10	2	1

Thus $p = 2$. If $B = 1$ then $u = 2$. And if $B = 0$ then $u = 1$.

On the other hand, $\mathcal{Z}^* = 2\mu - 2$. Then we obtain 1) $t_2 + t_\mu = 2$ or 2) $\mathcal{Z}^* = 2\mu - 2 \geq 3\mu - 9$ for $\mu \geq 6$. Hence, $\mu = 6$ or 7.

In the case when 1) $t_2 + t_\mu = 2$,by

$$\overline{Y} = 8\mu + 2k + 2 = t_2 + (\mu - 1)t_\mu + (r-2)\mu = 2 + (\mu - 2)t_\mu + (r-2)\mu,$$

we obtain $(\rho - 1)\mu = 2k - (\mu - 2)t_\mu$ where $t_\mu \leq 2$.

$t_\mu = 2$.

Then $(\rho - 2)\mu = 2k - 2(\mu - 2)$ and $\rho\mu = 2k + 4 = 24$.

Table 32:

ρ	r	μ	ν_1	σ	u	e
1	9	24	25	52	2	79
2	10	12	13	28	2	43
3	11	8	9	20	2	31
4	12	6	7	16	2	25
6	14	4	5	12	2	19
8	16	3	4	10	2	16
12	20	2	3	8	2	13

Table 33:

ρ	r	μ	ν_1	σ	u	e
1	9	24	25	52	1	53
2	10	12	13	28	1	29
3	11	8	9	20	1	21
4	12	6	7	16	1	17
6	14	4	5	12	1	13
8	16	3	4	10	1	11
12	20	2	3	8	1	9

$t_\mu = 1$.
Then $(\rho-2)\mu = 2k - \mu + 2$ and $(\rho-1)\mu = 2k + 2 = 22$.

$t_\mu = 0$.
Then $(\rho-2)\mu = 2k = 20$.

7.6.4 case when $\tilde{q} = 0$

Assume $\tilde{q} = 0$. Then

$$\mathcal{Z}^* = 2\mu.$$

If $\mu \geq 6$ then $2\mu = \mathcal{Z}^* \geq 3\mu - 9$; thus $\mu \leq 9$.

$\mu = 9$
Then

$$\mathcal{Z}^* = 2\mu = 18 = 8y_1 + 14y_2 + 18y_3.$$

Hence, $y_3 = t_4 + t_7 = 1$.

$$\overline{Y} = 8\mu + 2k + 2 = 3t_4 + 6t_7 + (r-1)\mu.$$

Thus, $9(\rho - 1) = (\rho - 1)\mu = 2k - 1 - 3t_7$.

But if $k = 8$ then $2k - 1 - 3t_7 = 15 - 3t_7$,which is not divisible by 9.

If $k = 9$ then $2k - 1 - 3t_7 = 17 - 3t_7$,which is not divisible by 9.

$\mu = 8$

Then

$$\mathcal{Z}^* = 2\mu = 16 = 7y_1 + 12y_2 + 15y_3 + 16y_4.$$

Hence, $y_4 = t_5 = 1$.

$$\overline{Y} = 8\mu + 2k + 2 = 4 + (r-1)\mu.$$

Thus, $8(\rho - 1) = (\rho - 1)\mu = 2k - 2$.

If $k = 9$ then $2k - 2 = 16$ and $\rho - 1 = 1$.

$\sigma = 21, e = 30, r = 11$ and the type turns out to be $[21 * 30, 1, ; 9^{10}, 5]$.

$\mu = 7$

Then

$$\mathcal{Z}^* = 2\mu = 14 = 6y_1 + 10y_2 + 12y_3.$$

No solution.

$\mu = 6$

Then

$$\mathcal{Z}^* = 2\mu = 12 = 5y_1 + 8y_2 + 9y_3.$$

No solution.

$\mu = 5$

Then

$$\mathcal{Z}^* = 2\mu = 10 = 4y_1 + 6y_2.$$

Then $y_1 = t_2 + t_5 = y_2 = t_3 + t_4 = 1$.

Hence,

$$5(\rho - 2) = 2k + 2 - 3 - t_4 - 3t_5.$$

If $k = 8$ then $2k + 2 - 3 - t_4 - 3t_5 = 15 - t_4 - 3t_5$. This is divisible by 5 whenever $t_4 = t_5 = 0$.

$\sigma = 14, u = 1, e = 21, r = 13$ and the type turns out to be $[14 * 21, 1, ; 6^{11}, 3, 2]$.

If $k = 9$ then there are no solution.

$\mu = 4$
Then
$$\mathcal{Z}^* = 2\mu = 8 = 3y_1 + 4y_2.$$
Then $y_2 = t_3 = 2$.
Hence,
$$4(\rho - 2) = 2k - 2.$$
Then $2(\rho - 2) = k - 1 = 8$ and $\rho = 6$.
$\sigma = 13, u = 0, e = 18, r = 14$ and the type turns out to be $[13 * 18, 1, ; 5^{12}, 3^2]$.

$\mu = 3$
Then
$$\mathcal{Z}^* = 2\mu = 6 = 2y_1.$$
Then $y_1 = t_2 + t_3 = 2$.
Hence,
$$3(\rho - 3) = 2k - 1 - t_3.$$
Then$k = 8, t_3 = 0$ and $\rho = 5$.
$\sigma = 10, u = 1, e = 15, r = 16$ and the type turns out to be $[10 * 15, 1, ; 4^{13}, 2^3]$.

$\mu = 2$
Then
$$\mathcal{Z}^* = 2\mu = 4 = y_1.$$
Then $y_1 = t_2 = .$
Hence,
$$\rho - 4 = k - 1.$$
If $k = 8$ then $\rho = 11, u = 1, p = 2$.
$\sigma = 8, u = 1, e = 12, r = 19$ and the type turns out to be $[8{*}12, 1, ; 3^{15}, 2^4]$.
If $k = 9$ then $\rho = 20, u = 0, p = 3$.
$\sigma = 9, u = 0, e = 12, r = 20$ and the type turns out to be $[9{*}12, 1, ; 3^{16}, 2^4]$.

7.7 case when $\mathcal{Z}^* = 0$

Suppose that $\mathcal{Z}^* = 0$. Then $0 = \mathcal{Z}^* = 2\mu - \tilde{q}$. Hence, $2\mu = \tilde{q} = k(p-1) - 2p^2$. Thus

$$2\mu + 2 = k(p-1) - 2p^2. \tag{25}$$

By $\overline{Y} = 8\mu + 2k + 2 = r\mu$, we obtain

$$\rho\mu = 2k + 2. \tag{26}$$

Thus

$$k(\rho p - \rho - 4) = 4 + 2\rho p^2 \tag{27}$$

7.7.1 case when $\rho = 1$

Suppose that $\rho = 1$. Then $k(p-5) = 4 + 2p^2$; thus

$$k = 2(p+5) + \frac{54}{p-5}. \tag{28}$$

Table 34: $B = 1, \rho = 1$

$p-5$	p	$54/(p-5)$	$2(p+5)$	k	u	μ	ν_1	σ	e
1	6	54	22	76	29	154	155	316	500
2	7	27	24	51	15	104	105	217	337
3	8	18	26	44	10	90	91	190	291
6	11	9	32	41	4	84	85	181	270
9	14	6	38	44	1	90	91	196	288

The types are as follows

1. $[316 * 500, 1; 155^9]$,
2. $[217 * 337, 1; 105^9]$,
3. $[190 * 291, 1; 91^9]$,
4. $[181 * 270, 1; 85^9]$,
5. $[196 * 288, 1; 91^9]$,

The types are as follows

1. $[316 * 342; 155^9]$,
2. $[190 * 196; 91^9]$,

Table 35: $B = 0, \rho = 1$

$p-5$	p	$p+5$	$54/(p-5)$	k	$k-4p$	u	μ	ν_1	σ	e
1	6	11	54	76	52	26	154	155	316	342
3	8	13	18	44	12	6	90	91	190	196

7.7.2 case when $\rho = 2$

Suppose that $\rho = 2$. Then $k(p-3) = 4 + 2p^2$; thus

$$k = 2(p+3) + \frac{20}{p-3}. \tag{29}$$

Table 36: $B = 1, \rho = 2$

$p-3$	p	$p+3$	20	k	$k-3p$	u	μ	ν_1	σ	e
1	4	7	20	34	22	11	35	36	76	123
4	7	10	5	25	4	2	26	27	61	90
5	8	11	4	26	2	1	27	28	64	93

The types are as follows

1. $[76 * 123, 1; 36^{10}]$,
2. $[61 * 90, 1; 27^{10}]$,
3. $[64 * 93, 1; 28^{10}]$,

Table 37: $B = 0, \rho = 2$

$p-3$	p	$p+3$	20	k	$k-4p$	u	μ	ν_1	σ	e
1	4	7	20	34	18	9	35	36	76	85
2	5	8	10	26	6	3	27	28	61	64

The types are as follows

1. $[76 * 85; 36^{10}]$,
2. $[61 * 64; 27^{10}]$.

7.7.3 case when $\rho = 3$

Suppose that $\rho = 3$. Then $k(3p-7) = 4 + 6p^2$; thus

$$k = 2(p+2) + \frac{2p+32}{3p-7}. \tag{30}$$

Hence, $p > 2$. Letting $m = \frac{2p+32}{3p-7}$, we obtain $Pm = 2p + 32$, where $P = 3p - 7$. Then $(3m-2)P = 110$. From this , the next tables follow.

Table 38: $\rho = 3, B = 1$

$3m-2$	$P = 3p-7$	p	m	k	μ	ν_1	σ	u	e
10	11	6	4	20	14	15	36	1	52
22	5	4	8	20	14	15	34	4	53
55	2	3	19	29	20	21	45	10	76

Table 39: $\rho = 3, B = 0$

$3m-2$	$P = 3p-7$	p	m	k	μ	ν_1	σ	u	e
22	5	4	8	20	14	15	34	2	36

The types are as follows

1. $[36 * 52, 1; 15^{11}]$,
2. $[34 * 53, 1; 15^{11}]$,
3. $[45 * 76, 1; 21^{11}]$,
4. $[34 * 36; 15^{11}]$.

7.7.4 case when $\rho = 4$

Suppose that $\rho = 4$. Then $k(p-2) = 1 + 2p^2$; thus

$$k = 2(p+2) + \frac{9}{p-2}. \tag{31}$$

Thus $p - 2 = 1, 3, 9$.
The types are as follows

Table 40: $\rho = 4, B = 1$

$p-2$	p	$2p+4$	$9/(p-2)$	k	$2k+2$	μ	ν_1	σ	u	e
1	3	10	9	19	40	10	11	25	5	41
3	5	14	3	17	36	9	10	25	1	36

1. $[25 * 41; 11^{12}]$,
2. $[25 * 36; 10^{12}]$.

7.7.5 case when $\rho \geq 5$

Since $k \geq 3p$, it follows that

$$k(\rho p - \rho - 4) = 4 + 2\rho p^2 \geq 3(\rho p - \rho - 4)p.$$

Hence,

$$12p + 4 + 3\rho p \geq \rho p^2.$$

Defining a function $f(x, y)$ to be $yx^2 - 12x - 4 - 3yx$, we get $f(5,5) = -14, f(6,5) = 14$. Therefore, we see that if $y \geq 5$ and $x \geq 6$, then $f(x,y) \geq 0$.

Therefore, if $\rho \geq 5$ then $5 \leq 4$. We shall enumerate all types with $p \leq 5$.

$p = 2$
By (27), we obtain $k(\rho - 4) = 4 + 8\rho$. Hence,

$$(k-8)(\rho-4) = 36.$$

From this we obtain the following table.

Table 41: $p = 2, B = 1$

$k-8$	$\rho-4$	k	ρ	$2k+2$	μ	ν_1	σ	u	e
4	9	12	13	26	2	3	8	3	14
6	6	14	10	30	3	4	10	4	18
12	3	20	7	42	6	7	16	7	30
18	2	26	6	54	9	10	22	10	42
36	1	44	5	90	18	19	40	19	78

Table 42: $p = 2, B = 0$

$k-8$	$\rho-4$	k	ρ	$2k+2$	μ	ν_1	σ	u	e
4	9	12	13	26	2	3	8	2	10
6	6	14	10	30	3	4	10	3	13
12	3	20	7	42	6	7	16	6	22
18	2	26	6	54	9	10	22	9	31
36	1	44	5	90	18	19	40	18	58

The types are as follows

1. $[8*14,1;3^{21}]$,
2. $[10*18,1;4^{18}]$,
3. $[16*30,1;6^{15}]$,
4. $[22*42,1;10^{114}]$,
5. $[40*78,1;19^{13}]$.

1. $[8*10;3^{21}]$,
2. $[10*13;4^{18}]$,
3. $[16*22;6^{15}]$,
4. $[22*31;10^{114}]$,
5. $[40*58;19^{13}]$.

$p = 3$
By (27), we obtain $k(\rho - 2) = 2 + 9\rho$. Hence,

$$(k-9)(\rho-2) = 20.$$

From this we obtain the following table.

Table 43: $p = 3, B = 1$

$k-9$	$\rho-2$	k	ρ	$2k+2$	μ	ν_1	σ	u	e
2	10	11	12	24	2	3	9	1	13
4	5	13	7	28	4	5	13	2	20
10	2	19	4	40	10	11	25	5	41
20	1	29	3	60	20	21	45	10	76

Table 44: $p = 3, B = 0$

$k-9$	$\rho-2$	k	ρ	$2k+2$	μ	ν_1	σ	u	e
5	4	14	6	30	5	6	15	1	16

The types are as follows

1. $[9 * 13, 1; 3^{20}]$,
2. $[13 * 20, 1; 5^{15}]$,
3. $[15 * 16; 6^{14}]$,
4. $[25 * 41, 1; 11^{12}]$,
5. $[45 * 76, 1; 21^{11}]$.

$p = 4$

By (27), we obtain $k(3\rho - 4) = 4 + 32\rho$. Hence, letting $K = 3k - 32, R = 3\rho - 4$, we get

$$KR = (3k - 32)(3\rho - 4) = 140.$$

From this we obtain the following table.

Table 45: $p = 4, B = 1$

K	R	$\rho - 2$	k	ρ	$2k + 2$	μ	ν_1	σ	u	e
4	35	12	13	26	2	3	10	0	13	
10	14	14	6	30	5	6	16	1	23	
28	5	20	3	42	14	15	34	4	53	
70	2	34	2	70	35	36	76	11	123	

Table 46: $p = 4, B = 0$

K	R	$\rho - 2$	k	ρ	$2k + 2$	μ	ν_1	σ	u	e
28	5	20	3	42	14	15	34	2	36	
70	2	34	2	70	35	36	76	9	85	

The types are as follows

1. $[10 * 13, 1; 3^{21}]$,
2. $[16 * 23, 1; 6^{14}]$,
3. $[34 * 53, 1; 15^{11}]$,
4. $[76 * 123, 1; 36^{10}]$.
5. $[34 * 36; 15^{11}]$,
6. $[76 * 85; 36^{10}]$.

$p = 5$

By (27), we obtain $k(4\rho - 4) = 4 + 50\rho$. Hence, letting $K = 2k - 25, R = \rho - 1$, we get

$$KR = (2k - 25)(\rho - 1) = 27.$$

From this we obtain the following table.

Table 47: $p = 5$

K	R	ρ	k	μ	ν_1
1	27	28	13	1	2
3	9	10	14	3	4
9	3	4	17	9	10
27	1	2	26	27	28

If $B = 1$ then $u = (k - 3p)/2$ is integer; thus the case when $\rho = 4, k = 17, \mu = 9$ survives. Indeed, $u = (15 - 3 \times 5)/2 = 1$. Hence, $\sigma = 25, e = 36, r = 12$. The type turns out to be $[25 * 36, 1; 20^{12}]$.

If $B = 0$ then $u = (k - 4p)/2$ is integer; thus the case when $\rho = 2, k = 26, \mu 27$ survives. The type turns out to be $[61 * 64; 28^{10}]$.

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