On the birational invariants k and genus of algebraic plane curves

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1 Introduction

Here, we shall study birational properties of algebraic plane curves from the viewpoint of Cremonian geometry. As a matter of fact, let S be a nonsingular rational surface and D a nonsingular curve on S. (S, D) are called pairs and we study such pairs. The purpose of Cremonian geometry is the study of birational properties of pairs (S, D).

Suppose that $m \geq a \geq 1$. Then $P_{m,a}[D] = \dim |mK_S + aD| + 1$ are called mixed plurigenera, which depend on S and D. It is my understanding that these invariants embody the essential geometric properties of the curve D on S. $P_{1,1}[D]$ turns out to be the genus of D, denoted by g.

Letting Z stand for $K_S + D$, we see $P_{m,m}[D] = \dim |mZ| + 1$, called *logarithmic plurigenera* of $S - D$, from which logarithmic Kodaira dimension is introduced, denoted by $\kappa[D]$.

Assume that $\kappa[D] = 2$ and that there exist no (-1) curves E such that $E \cdot D \leq 1$. Then such pairs are proved to be minimal in the birational geometry of pairs ([7],[6]).

We start with recalling some basic results in birational geometry of pairs (S, D).

Minimal pairs are obtained from some kind of singular models, namely, # minimal pairs which will be defined below. Any nontrivial $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$ has a section Δ_∞ with negative self intersection number, which is denoted by a symbol Σ_B, where $-B = {\Delta_\infty}^2$ if $B > 0$. Σ_B is said to be a Hirzebruch surface of degree B after Kodaira.

Let Σ_0 denote the product of two projective lines.

The Picard group of Σ_B is generated by a section Δ_∞ and a fiber $F_c = pr^{-1}(c)$ of the $\mathbf{P}^1-$ bundle, where $c \in \mathbf{P}^1$ and $pr : \Sigma_B \to \mathbf{P}^1$ is the projection.

Let C be an irreducible curve on Σ_B. Then $C \sim \sigma\Delta_\infty + eF_c$, for some integers σ and e. Here the symbol $\sim$ means the linear equivalence between divisors. We have $C \cdot F_c = \sigma$ and $C \cdot \Delta_\infty = e - B \cdot \sigma$.

Note that $\kappa[\Delta_\infty] = -\infty$.

Hereafter, suppose that $C \neq \Delta_\infty$. Thus $C \cdot \Delta_\infty = e - B \cdot \sigma \geq 0$ and hence, $e \geq B\sigma$. Denoting $2e - B\sigma$ by $\widetilde{B}$, we have the formula of the virtual genus of C denoted by g_0:

$$g_0 = \frac{(\sigma - 1)(\widetilde{B} - 2)}{2}.$$

Thus introducing τ_m by

$$\tau_m = (\sigma - m)(\widetilde{B} - 2m), \tag{1}$$

we obtain

$$(K_0 + C)^2 = \tau_2,$$

where K_0 denotes a canonical divisor on Σ_B.

Moreover, letting Z_0 be $K_0 + C$, we obtain for $\nu > 0$,

$$\nu Z_0 - (\nu - 1)C \sim C + \nu K_0$$

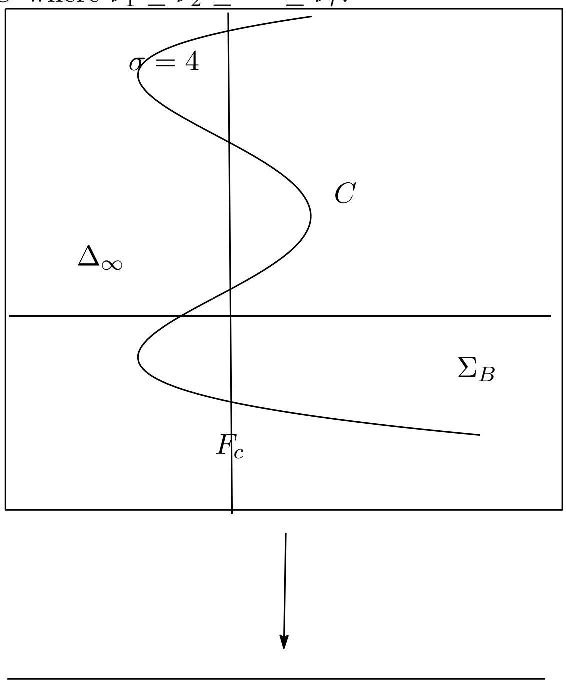
$$(\nu Z_0 - (\nu - 1)C) \cdot Z_0 = \tau_{\nu+1} - 2(\nu - 1)^2,$$

and

$$(\nu Z_0 - (\nu - 1)C) \cdot C = \tau_\nu - 2\nu^2.$$

1.1 minimal models

Let C be an irreducible curve on Σ_B. Then by $\nu_1, \nu_2, \cdots, \nu_r$ we denote the multiplicities of all singular points (including infinitely near singular points) of C where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.



The symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ is said to be the **type** of (Σ_B, C). When $B = 0$,the symbol is abbreviated as $[\sigma * e; \nu_1, \nu_2, \cdots, \nu_r]$.

Definition 1 *The pair (Σ_B, C) is said to be* # ***minimal*** *, if*

- $\sigma \geq 2\nu_1$ *and* $e - \sigma \geq B\nu_1$.
- *Moreover, if* $B = 1$ *and* $r = 0$ *then assume* $e - \sigma > 1$.

Using elementary transformations, we get

Theorem 1 *If D is not transformed into a line on $\mathbf{P^2}$ by Cremona transformations, then $\kappa[D] \geq 0$. In this case, a minimal pair (S, D) is obtained* *from a # minimal pair (Σ_B, C) by shortest resolution of singularities of C using blowing ups except for $(S, D) = (\mathbf{P^2}, C_d)$, C_d being a nonsingular curve of degree $d > 2$.*

Theorem 2 *If (S, D) is obtained from a # minimal pair (Σ_B, C) by shortest resolution of singularities of C ,then (S, D) is relatively minimal. In other words,for any (-1) curve Γ on S, $\Gamma \cdot \Delta \geq 2$.*

2 basic results

Suppose that (S, D) is a minimal pair with $\kappa[D] = 2$, which is obtained from a # minimal pair(model) (Σ_B, C) by shortest resolution of singularities of C. The type of (Σ_B, C) is denoted by the symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$.

By

$$2\omega = (D + 3K_S) \cdot D, 2\omega_0 = (C + 3K_0) \cdot C,$$

and

$$2\overline{g} = (D + K_S) \cdot D, 2\overline{g}_0 = (C + K_0) \cdot C,$$

we get

- $\omega = \omega_0 - \sum_{j=1}^{r} \dfrac{\nu_j(\nu_j - 3)}{2}$,
- $g = g_0 - \sum_{j=1}^{r} \dfrac{\nu_j(\nu_j - 1)}{2}$.

By putting $X = \sum_{j=1}^{r} \nu_j^2$ and $Y = \sum_{j=1}^{r} \nu_j$, we obtain

- $2\omega - 2\omega_0 = -X + 3Y$,
- $2g - g_0 = -X + Y$.

Thus

- $X = 3g_0 - \omega_0 - 3g + \omega$,
- $Y = g_0 - \omega_0 - g + \omega$.

However, from $\omega_0 = \frac{\tau_3}{2} - 9$ and $\overline{g_0} = g_0 - 1 = \frac{\tau_1}{2} - 1$, it follows that

- $\overline{g_0} - \omega_0 = \widetilde{B} + 2\sigma$,
- $3\overline{g_0} - \omega_0 = \widetilde{B}\sigma$.

Consequently we obtain the next equalities:

- $Y = \widetilde{B} + 2\sigma + \omega - \overline{g}$,
- $X = \widetilde{B}\sigma + \omega - 3\overline{g}$.

2.1 two invariants

We shall compute two invariants $\widetilde{B} + 2\sigma$ and $\widetilde{B}\sigma$ by examining the following cases according to the value of B.

(1) $B = 0$. Then $\sigma = 2\nu_1 + p, e = \sigma + u$ for some $u \geq 0$ and

- $\widetilde{B} + 2\sigma = 8\nu_1 + 4p + 2u$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2\nu_1(4p + 2u) + 2pu + 2p^2$.

(2) case $B = 1$. Then $\sigma = 2\nu_1 + p, e = \sigma + \nu_1 + u$ for some $u \geq 0$ and

- $\widetilde{B} + 2\sigma = 8\nu_1 + 3p + 2u$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2\nu_1(3p + 2u) + 2pu + p^2$.

(3) $B = 2$. Then $\sigma = 2\nu_1 + p, e = 2\sigma + u$ for some $u \geq 0$ and

- $\widetilde{B} + 2\sigma = 8\nu_1 + 4p + 2u$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2\nu_1(4p + 2u) + 2pu + 2p^2$.

Defining $w = 4 - \delta_{1B}$,we get $w = 4$ if $B \neq 1$. Further, $w = 3$ if $B = 1$. Introducing an invariant k by $k = wp + 2u$, we have

- $\widetilde{B} + 2\sigma = 8\nu_1 + k$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2k\nu_1 + p(k - 2p)$.

Proposition 1 *Suppose that $B \leq 2$. Letting k denote $wp + 2u$, w being $4 - \delta_{1B}$, we have the following fundamental equalities:*

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + k + \omega_1$.

Here $\tilde{k} = kp - 2p^2, \omega_1 = \omega - \overline{g}$.

2.2 invariant $\widetilde{\mathcal{Z}}$

Following Matsuda([13]), we shall compute $\nu_1 Y - X$, which we denote by $\widetilde{\mathcal{Z}}$.

By $\widetilde{\mathcal{Z}} = \nu_1 Y - X = \sum_{j=1}^{r} \nu_j(\nu_1 - \nu_j) \geq 0$, we have

$$0 \leq \widetilde{\mathcal{Z}} = \nu_1(\omega - \overline{g} - k) - \tilde{k} - \omega_1 + 2\overline{g}. \tag{2}$$

2.3 case in which $B \geq 3$

By B_2 we denote $\max\{B-2, 0\}$. Then $e = B\sigma + u = B_2\sigma + 2\sigma + u$ for some $u \geq 0$ and $\widetilde{B} = 2e - B\sigma = B_2\sigma + 2(\sigma + u)$.

Moreover, $\widetilde{B}\sigma = B_2\sigma^2 + 2(\sigma + u)\sigma$ and so

- $\widetilde{B} + 2\sigma = B_2\sigma + 8\nu_1 + k$,
- $\widetilde{B}\sigma = B_2\sigma^2 + 8\nu_1^2 + 2k\nu_1 + \tilde{k}$.

However,these formulas still hold for any $B \geq 0$. Thus, we obtain the following fundamental equalities:

- $X = B_2\sigma^2 + 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = B_2\sigma + 8\nu_1 + k + \omega_1$.

Further , we get

$$0 \leq \widetilde{\mathcal{Z}} = B_2\sigma(\nu_1 - \sigma) - k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} - \tilde{k},$$

and

$$B_2\sigma(\sigma - \nu_1) \leq -k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} - \tilde{k}.$$

If $B \geq 3$, then

$$\sigma(\sigma - \nu_1) \leq B_2\sigma(\sigma - \nu_1) \leq -k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} - \tilde{k}. \tag{3}$$

Hence, the following is derived:

Proposition 2 *If $B \geq 3$,then*

$$2\nu_1^2 \leq \sigma(\sigma - \nu_1) \leq (\nu_1 - 1)\omega_1 + 2\overline{g}.$$

3 estimate of k in terms of ω

We shall prove the following estimate of k.

Proposition 3 *If $\sigma \geq 7$ and $\nu_1 \geq 3$,then $k \leq \omega$. Moreover, if $g > 0$, then $k \leq \omega - 1$. Assume $k = \omega$. Then types are as follows:*

In the case where $p = 0$, the type becomes $[10 * 11; 5^9]$ or its associates.

In the case where $p = 1$, the type becomes either 1) $[(4k+3) * (6k+u+4), 1; (2k+1)^9]$,where $k = 3+2u, u \geq 0$, or 2) $[(19+8u)*(19+9u); (9+4u)^9]$, where $u \geq 0$

In the case where $p > 1$, $p = 2$ and the type becomes $[28 * 41, 1; 13^9]$.

Proof.

First , we shall prove $k \leq \omega$. From the following fundamental equalities: we see that $\widetilde{\mathcal{Z}} = \nu_1 Y - X \geq 0$ satisfies

$$0 \leq \widetilde{\mathcal{Z}} = B_2(\nu_1 - \sigma)\sigma - k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g},$$

and hence

$$\begin{aligned} 0 \leq B_2(\sigma - \nu_1)\sigma &\leq -k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g} \\ &\leq -k\nu_1 + \nu_1\omega - \omega + \overline{g}(3 - \nu_1). \end{aligned}$$

Thus when $\overline{g} \geq 0$, we get

$$k\nu_1 \leq \nu_1\omega - \omega.$$

Hence,

$$k \leq \omega - \frac{\omega}{\nu_1} < \omega.$$

However, when $\overline{g} = -1$, we get

$$k\nu_1 \leq \nu_1\omega - \omega + \nu_1 - 3.$$

Hence,

$$k - \omega \leq 1 - \frac{3 + \omega}{\nu_1} < 1.$$

Therefore, $k \leq \omega$,since $k - \omega$ is an integer.

3.1 the invariant i

Assume $\nu_1 \geq 3$. Introducing an **invariant** i by $i = \omega - k \geq 0$. we shall enumerate types whenever $i \leq 2$.

First, we shall prove that $B \leq 2$. Otherwise, we have $B_2 > 0$ and so

$$B_2(\sigma - \nu_1)\sigma \geq 2{\nu_1}^2 \geq 6\nu_1.$$

From

$$\begin{aligned} 6\nu_1 \leq B_2(\sigma - \nu_1)\sigma &\leq -k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g} \\ &\leq -k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega + \nu_1 - 3 \\ &= (\omega - k + 1)\nu_1 - \tilde{k} - \omega - 3 \\ &= (i + 1)\nu_1 - \tilde{k} - k + i - 3 \\ &\leq (i + 1)\nu_1 + i - 3, \end{aligned}$$

it follows that

$$5\nu_1 + 3 \leq i(\nu_1 + 1).$$

Hence,

$$4 \leq \frac{5\nu_1 + 3}{\nu_1 + 1} \leq i.$$

This contradicts the hypothesis saying $i \leq 2$.

3.2 case when $k = \omega$

Assume $i = 0$, i.e. $k = \omega$ and by the previous argument, $\overline{g} = -1$.

Supposing that $\widetilde{Z} > 0$, we get $\widetilde{Z} \geq \nu_1 - 1$. Hence,

$$\begin{aligned} \nu_1 - 1 \leq \widetilde{\mathcal{Z}} &\leq -k\nu_1 + (\nu_1 - 1)k + \overline{g}(3 - \nu_1) - \tilde{k} \\ &= -k - (3 - \nu_1) - \tilde{k} \\ &\leq \nu_1 - 3. \end{aligned}$$

Thus $\nu_1 - 1 \leq \nu_1 - 3$, which is a contradiction. Therefore, $\widetilde{Z} = 0$.

3.3 a formula for i

In general, in the case when $B \leq 2, \overline{g} = -1$ and $\widetilde{Z} = 0$, we obtain the following formulae from the fundamental equalities :

- $\omega_1 = i + 1 + k$,
- $(r-8)\nu_1 = k + \omega_1 = 2k + i + 1$,
- $(r-8)\nu_1^2 = 2k\nu_1 + \omega_1 + \tilde{k} + 2 = 2k\nu_1 + \tilde{k} + i + k + 3$.

Then $r \geq 9$ and

$$\nu_1 = \frac{2k+i+1}{r-8}. \tag{4}$$

Introducing ρ by $\rho = r - 8$, these are rewritten as follows:

1. $\rho\nu_1 = k + \omega_1 = 2k + i + 1$,
2. $\rho\nu_1^2 = 2k\nu_1 + \omega_1 + \tilde{k} + 2 = 2k\nu_1 + \tilde{k} + i + k + 3$,
3. $\rho = r - 8 \geq 1$,
4. $\rho\nu_1 = 2k + i + 1$.

Thus,the formulae (1) and (2) yield

$$(i+1)\rho\nu_1 = \rho(\tilde{k} + i + k + 3).$$

By (1) we obtain

$$(i+1)(2k+i+1) = \rho(\tilde{k} + i + k + 3), \tag{5}$$

and

$$k(2i + 2 - \rho) + (i+1)^2 = \rho(\tilde{k} + i + 3). \tag{6}$$

3.4 case in which $i = 0$

Suppose that $i = 0$. From the formula (6), it follows that

$$k(2-\rho) + 1 = \tilde{k} + 3.$$

Hence, $\rho = 1$; $r = 9$ and $k + 1 = \tilde{k} + 3$; $k = \tilde{k} + 2$.

Therefore, from $k = \tilde{k} + 2 = p(k-2p) + 2$, it follows that

$$2p^2 - 2 = k(p-1).$$

Hence,we get either 1) $p = 1$ or 2) $p \neq 1$; $k = 2p + 2$.

In the case when $p = 1$, we have $\nu_1 = 2k + 1$ and $k = w + 2u$, where $w = 4 - \delta_{1B}$.

If $B = 1$ then $k = 3 + 2u$ and $\sigma = 2\nu_1 + p = 4k + 3$; $e = \sigma + \nu_1 + u = 6k + u + 4$. Thus the type becomes $[(4k+3) * (6k+u+4), 1; (2k+1)^9]$, where $k = 3 + 2u$.

Conversely, if the minimal pair (S, D) has this type, then

$g = \frac{(\sigma-1)(\widetilde{B}-2)}{2} - 9(2k+1)k = 0$ and $D^2 = \sigma\widetilde{B} - 9(2k+1)^2 = -k - 3$. Thus $\omega = -3 - (-k-3) = k$.

If $B = 0$ then $k = 4 + 2u$ and $\nu_1 = 2k + 1 = 9 + 4u, \sigma = 2\nu_1 + p = 4k + 3 = 19 + 8u$. Thus $e = \sigma + u = 19 + 9u$ and the type becomes $[(19 + 8u) * (19 + 9u); (9 + 4u)^9]$.

Conversely, if the minimal pair (S, D) has this type, then $g = 0$ and $\omega = 4 + 2u = k$.

In the case when $k = 2p + 2$ and $p \neq 1$, we have either $p = 0$ or $p > 1$.

If $p = 0$ then $u = 1$ and $k = 2$. Thus $\nu_1 = 2k + 1 = 5, \sigma = 10$ and $B \leq 2$.

If $B = 0$ then the type becomes $[10 * 11; 5^9]$.

If $B = 1$ then the type becomes $[10 * 16, 1; 5^9]$.

If $B = 2$ then the type becomes $[10 * 21, 2; 5^9]$ or its associates.

Note : The types $[10 * 16, 1; 5^9]$ and $[10 * 21, 2; 5^9]$ are said to be the associates of the type $[10*11; 5^9]$. Hereafter, such associates will be omitted, for simplicity.

If $p > 1$ then $k = 2p+2 = wp+2u$, from which it follows that $p = 2, u = 0, w = 3, k = 6$ and $B = 1$.

Moreover, $\nu_1 = 2k + 1 = 13$ and $\sigma = 28$ and $e = 41$. Hence, the type becomes $[28 * 41, 1; 13^9]$.

Conversely, if the minimal pair (S, D) has this type, then $g = 0$ and $D^2 = -9$ and $\omega = 6 = k$.

Therefore, the proof of Proposition 1 is complete. In that follows we shall enumerate all possible types whenever $i = 1$ or 2.

4 formula (FEQ)

Suppose that $B \leq 2$ and that a $\#-$ minimal pair (Σ_B, C) has j_1 singular points with multiplicity $\nu_1 - 1$ and j_2 singular points with multiplicity $\nu_1 - 2$. Moreover, assume that the other singular points have the multiplicity ν_1. Then

- $Y = \nu_1(r - j_1 - j_2) + j_1(\nu_1 - 1) + j_2(\nu_1 - 2) = r\nu_1 - j_1 - 2j_2$,
- $X = \nu_1^2(r - j_1 - j_2) + j_1(\nu_1 - 1)^2 + j_2(\nu_1 - 2)^2 = r\nu_1^2 - 2j_1\nu_1 - 4j_2\nu_1 + j_1 + 4j_2$.

From the fundamental equalities, we obtain

$$\begin{aligned}\rho\nu_1 &= j_1 + 2j_2 + k + \omega_1,\\ &= j_1 + 2j_2 + 2k + i - \overline{g}\end{aligned}$$

and

$$\begin{aligned}\rho\nu_1^2 &= 2j_1\nu_1 + 4j_2 - j_1 - 4j_2 + 2k\nu_1 + \tilde{k} + k + i - 3\overline{g}\\ &= j_1 + 2j_2 + k + \omega - \overline{g} + \tilde{k} + \omega_1 - 2\overline{g}.\end{aligned}$$

But from

$$\rho\nu_1^2 = (j_1 + 2j_2 + 2k + i - \overline{g})\nu_1$$

it follows that

$$k(2i - 2j_1 - 4j_2 - 2\overline{g} - \rho) + (\overline{g} - i)^2 - (j_1 + 2j_2)^2 = \rho(\tilde{k} + i - 3\overline{g} - j_1 - 4j_2). \quad (7)$$

This will be referred to as the formula (FEQ).

Proposition 4 *When $B \leq 2$ and a $\#-$ minimal pair (Σ_B, C) has j_1 singular points with multiplicity $\nu_1 - 1$ and j_2 singular points with multiplicity $\nu_1 - 2$, and the other singular points have the multiplicity ν_1, the next equalities hold.*

$$k(2i-2j_1-4j_2-2\overline{g}-\rho)+(\overline{g}-i)^2-(j_1+2j_2)^2=\rho(\tilde{k}+i-3\overline{g}-j_1-4j_2) \quad (8)$$

and

$$\rho\nu_1 = j_1 + 2j_2 + 2k + i - \overline{g}.$$

5 case when $j_1 = j_2 = 0$

Assuming $j_1 = j_2 = 0$, we have from (FEQ) the next equality:

$$k(2i - 2\overline{g} - \rho) + (\overline{g} - i)^2 = \rho(\tilde{k} + i - 3\overline{g}).$$

5.1 case when $g = 0$

Suppose that $g = 0$. Then

$$k(2i + 2 - \rho) + (1 + i)^2 = \rho(\tilde{k} + i + 3)$$

and

$$\rho\nu_1 = 2k + i + 1.$$

We shall study the types when $i = 1, 2$.

5.1.1 case when $i = 1$

If $i = 1$ then the formula (FEQ) turns out to be

$$k(4 - \rho) + 4 = \rho(\tilde{k} + 4).$$

Then $\rho = 1$ or 2 or 3.

i) Suppose that $\rho = 1$. Then $3k + 4 = \tilde{k} + 4$. Hence,

$$3k = \tilde{k} = p(k - 2p).$$

From $2p^2 = (p - 3)k$, it follows that

$$2(p + 3) + \frac{18}{p - 3} = k \geq 3p.$$

We obtain the next table.

Table 1: case when $g = 0, i = 1, \rho = 1, B = 1$

$p-3$	p	$2(p+3)$	$18/(p-3)$	k	ν_1	σ	e	$3p$	u
1	4	14	18	32	66	136	212	12	10
2	5	16	9	25	52	109	166	15	5
3	6	18	6	24	50	106	159	18	3
6	9	24	3	27	56	121	177	27	0

Conversely, if the type of the pair (S, D) is $[136 * 212, 1; 66^9]$,then $g = 0, \omega = 33, k = 32$.

If the type of the pair is $[109 * 166, 1; 52^9]$,then $g = 0, \omega = 26, k = 25$.

If the type of the pair is $[106 * 159, 1; 50^9]$,then $g = 0, \omega = 25, k = 24$.

If the type of the pair is $[121 * 177, 1; 56^9]$,then $g = 0, \omega = 28, k = 27$.

Table 2: case when $g = 0, i = 1, \rho = 1, B = 0$

$p-3$	p	$2(p+3)$	$18/(p-3)$	k	ν_1	σ	e	$4p$	u
1	4	14	18	32	66	136	144	16	8
3	6	18	6	24	50	106	106	24	0

Conversely, if the type of the pair (S, D) is $[136 * 144; 66^9]$,then $g = 0, \omega = 33, k = 32, Z^2 = 31$.

If the type of the pair (S, D) is $[106 * 106, 1; 50^9]$,then $g = 0, \omega = 25, k = 24$.

ii) Suppose that $\rho = 2$. Thus $\nu_1 = k + 1, k + 2 = \tilde{k} + 4$. Hence,

$$2p^2 - 2 = (p - 1)k.$$

a) If $p \neq 1$ then $2p + 2 = k = wp + 2u$, where $w = 3$ or 4.

If $B = 1$, we obtain $p = 2, k = 6, u = 0, \nu_1 = 7$. Then $\sigma = 2\nu_1 + p = 16, e = 16 + 7 = 23$. Thus the type is $[16 * 23, 1; 7^{10}]$. Conversely, if the type is this, then $\omega = 7, g = 0, k = 6$.

If $B = 0$, we obtain $p = 0, k = 2, u = 1$. Thus, $2\nu_1 = \rho\nu_1 = 2k + i - \overline{g} = 5 - \overline{g} \leq 6$. Hence, $\nu_1 = 3, \sigma = 6$. But $\sigma \geq 7$ was assumed.

b) If $p = 1$ then $\tilde{k} = k - 2$. But $B = 1$ or $B = 0$.

If $B = 1$, then $k = 3 + 2u$; $\nu_1 = k + 1$, $\sigma = 2\nu_1 + p = 9 + 4u, e = 13 + 7u$. Thus the type is $[(9 + 4u) * (13 + 7u), 1; (4 + 2u)^{10}]$.

Conversely, if the pair has this type, then $\omega = 4 + 2u, g = 0$ and $k = 3 + 2u$.

If $B = 0$, then $k = 4+2u$; $\nu_1 = k+1 = 5+2u$, $\sigma = 2\nu_1+p = 11+4u, e = 11 + 5u$. Thus the type is $[(11 + 4u) * (11 + 5u); (5 + 2u)^{10}]$.

Conversely, if the pair has this type, then $\omega = 5 + 2u, g = 0$ and $k = 4 + 2u$.

iii) Suppose that $\rho = 3$. Then $3\tilde{k} = k - 8$ and

$$k - 8 = 3\tilde{k} = 3(p(k - 2p)).$$

Hence,

$$6p^2 - 8 = (3p - 1)k \geq 3p(3p - 1) = 9p^2 - 3p.$$

From this it follows that

$$3p - 6 \geq 3p - 8 \geq 3p^2.$$

Hence, $p - 2 \geq p^2$. This is a contradiction.

5.1.2 case when $i = 2$

If $i = 2$ then the formula (FEQ) turns out to be

$$k(6 - \rho) + 9 = \rho(\tilde{k} + 5).$$

Since $\tilde{k} = p(k - 2p)$, it follows that

$$k(p\rho + \rho - 6) = 2p^2\rho - 5\rho + 9.$$

But recalling $k = wp + 2u \geq 3p$, we obtain

$$2p^2\rho - 5\rho + 9 = k(p\rho + \rho - 6) \geq 3p(p\rho + \rho - 6) = 3p^2\rho + 3p(\rho - 6).$$

Thus

$$-5\rho + 9 - 3p(\rho - 6) \geq p^2\rho.$$

Hence,

$$9 + 18p \geq \rho(p^2 + 3p + 5),$$

and

$$\frac{9 + 18p}{p^2 + 3p + 5} \geq \rho. \tag{9}$$

Therefore,

- if $p \leq 2$ then $\rho \leq 3$;
- if $3 \leq p \leq 5$ then $\rho \leq 2$;
- if $6 \leq p$ then $\rho = 1$.

Hence, $\rho \leq 3$.

i) Assume that $\rho = 1$. Then $r = 9$ and $\tilde{k}+5 = 5k+9$. From $\tilde{k} = p(k-2p)$, it follows that

$$2p^2 + 4 = k(p - 5). \tag{10}$$

Then $p > 5$ and we obtain

$$2(p + 5) + \frac{54}{p - 5} = k. \tag{11}$$

Then the following two tables are gotten.

Table 3: $g = 0, i = 2, B = 1$

$p-5$	p	$2(p+5)$	$54/(p-5)$	k	ν_1	σ	e	$3p$	u
1	6	22	54	76	155	316	500	18	29
2	7	24	27	51	105	217	337	21	15
3	8	26	18	44	91	190	291	24	10
6	11	32	9	41	85	181	270	33	4
9	14	38	6	44	91	196	288	42	1

Conversely, if the type is $[316 * 500, 1; 155^9]$,then $g = 0, \omega = 78, k = 76$.
If the type is $[217 * 337, 1; 105^9]$,then $g = 0, \omega = 53, k = 51$.
If the type is $[190 * 291, 1; 91^9]$,then $g = 0, \omega = 46, k = 44$.
If the type is $[181 * 270, 1; 85^9]$,then $g = 0, \omega = 43, k = 41$.
If the type is $[196 * 288, 1; 91^9]$,then $g = 0, \omega = 46, k = 44$.

Table 4: $g = 0, i = 2, B = 0$

$p-5$	p	$2(p+5)$	$54/(p-5)$	k	ν_1	σ	e	$4p$	u
1	6	22	54	76	155	316	342	24	26
3	8	26	18	44	91	190	196	32	6

Conversely, if the type is $[316{*}342; 155^9]$,then $g = 0, \omega = 78, k = 76, Z^2 = 76$.

If the type is $[190 * 196; 91^9]$,then $g = 0, \omega = 46, k = 44, Z^2 = 44$.

ii) Assume that $\rho = 2$. Then

$$4k + 9 = 2(\tilde{k} + 5).$$

Thus $9 = 2(\tilde{k} + 5) - 4k$, which is a contradiction.

iii) Assume that $\rho = 3$. Then $k + 3 = \tilde{k} + 5$; hence, $k = \tilde{k} + 2$ and

$$3\nu_1 = \rho\nu_1 = 2k + i + 1 = 2k + 3.$$

Then $k = \tilde{k} + 2 = p(k - 2p)$. ;thus, $(p - 1)k = 2(p^2 - 1)$.

a).If $p = 1$ then $\tilde{k} = k - 2$ and so $k = w + 2u$, where $w = 3$ or 4.

If $B = 1$ then $w = 3, k = 3 + 2u$ and

$$3\nu_1 = \rho\nu_1 = 2k + i + 1 = 2k + 3 = 9 + 4u.$$

From $3(\nu_1 - 3) = 4u$, it follows that $\nu_1 - 3 = 4L, u = 3L$, for some L.

Then $\sigma = 8L+7, e = 15L+10$ and the type is $[(8L+7){*}(15L+10), 1; (3+4L)^{11}]$.

Conversely, if the type of the pair (S, D) is this , then $g = 0, \omega = 5 + 6L, k = 3 + 6L$.

If $B = 1$ then $w = 4, k = 4 + 2u$ and $3\nu_1 = 2k + 3 = 11 + 4u$.

From $3(\nu_1 - 5) = 4(u - 1)$, it follows that $\nu_1 - 5 = 4L, u = 3L + 1$, for some L. Then $\sigma = 8L + 11, e = 11L + 12$ and the type is $[(8L + 11) * (11L + 12); (5 + 4L)^{11}]$.

Conversely, if the type of the pair (S, D) is this , then $g = 0, \omega = 8 + 6L, k = 6 + 6L$.

5.2 case when $g = 1$

Suppose that $g = 1$. Then

$$k(2i - \rho) + i^2 = \rho(\tilde{k} + i).$$

This implies that $i > 0$. We shall enumerate the types when $i = 1, 2$.

5.2.1 case when $i = 1$

If $i = 1$ then $k(2 - \rho) + 1 = \rho(\tilde{k} + 1)$. Thus $\rho = 1$, $\nu_1 = 2k + 1$ and $k + 1 = \tilde{k} + 1$. Hence,

$$k = \tilde{k} = p(k - 2p).$$

Thus

$$2p^2 - 2 + 2 = (p - 1)k,$$

and

$$2(p + 1) + \frac{2}{p - 1} = k$$

Hence, $p - 1 = 1$ or 2.

If $p = 2$ then $k = 8 = 2w + 2u$.
In the case when $B = 1$, we obtain $u = 1, \nu_1 = 2k+1 = 17, \sigma = 2\nu_1+p = 34 + 2 = 36$ and $e = 36 + 17 + 1 = 54$. Thus the type is $[36 * 54, 1; 17^9]$.
Conversely, if the type is this, then $\omega = 9, g = 1$.

In the case when $B = 0$, we obtain $u = 0, \nu_1 = 2k+1 = 17, \sigma = 2\nu_1+p = 34 + 2 = 36$ and $e = 36$. Thus the type is $[36 * 36; 17^9]$.
Conversely, if the type is this, then $\omega = 9, k = 8, g = 1$.

If $p = 3$ then $k = 9 = 3w+2u, w = 3, u = 0$ and the type is $[41*60, 1; 19^9]$.
Conversely, if the type is this, then $\omega = 10, k = 9, g = 1$.

5.2.2 case when $i = 2$

If $i = 2$ then

$$k(4 - \rho) + 4 = \rho(\tilde{k} + 2).$$

Then $\rho = 1$ or 2 or 3.

i) Assume that $\rho = 1$. Then $\nu_1 = 2k + 2$, $r = 9$ and $3k + 4 = \tilde{k} + 2$. From $\tilde{k} = p(k - 2p)$, it follows that

$$2p^2 + 2 = (p - 3)k. \tag{12}$$

Thus

$$2(p + 3) + \frac{20}{p - 3} = k.$$

Table 5: case when $\rho = 1$ and $g = 1, i = 2, B = 1$

$p - 3$	p	$2(p + 3)$	$20/(p - 3)$	k	ν_1	σ	e	$3p$	u
1	4	14	20	34	70	144	225	12	11
4	7	20	5	25	52	111	165	21	2
5	8	22	4	26	54	116	171	24	1
10	13	32	2	34	70	153	220.5	39	none

Conversely, if the type is $[144 * 225, 1; 70^9]$,then $g = 1, \omega = 36, k = 34$.
If the type is $[111 * 165, 1; 52^9]$,then $g = 1, \omega = 27, k = 25$.
If the type is $[116 * 171, 1; 54^9]$,then $g = 1, \omega = 26, k = 24$.

Table 6: case when $\rho = 1$ and $g = 1, i = 2, B = 0$

$p - 3$	p	$2(p + 3)$	$20/(p - 3)$	k	ν_1	σ	e	$4p$	u
1	4	14	20	34	70	144	153	16	9
2	5	16	10	26	54	113	116	20	3

Conversely, if the type is $[144 * 153; 70^9]$,then $g = 1, \omega = 36, k = 34$.
If the type is $[113 * 116; 54^9]$,then $g = 1, \omega = 28, k = 26$.

ii) Assume that $\rho = 2$. Then $\nu_1 = k + 1$, $r = 19$ and $2k + 4 = 2(\tilde{k} + 2)$. Hence, $k + 2 = \tilde{k} + 2$. From $\tilde{k} = p(k - 2p)$, it follows that

$$2p^2 = (p - 1)k. \tag{13}$$

Thus

$$2(p + 1) + \frac{2}{p - 1} = k.$$

Hence $p = 2$ or 3. If $p = 2$ then $k = 8, \nu_1 = 9$. In the case when $B = 1$, we obtain $u = 1, \nu_1 = k + 1 = 9$. Then $\sigma = 2\nu_1 + p = 18 + 2 = 20$ and $e = 20 + 9 + 1 = 30$. Thus the type becomes $[20 * 30, 1; 9^9]$.

In the case when $B = 0$, we obtain $u = 0, \nu_1 = k + 1 = 9$. Then $\sigma = 2\nu_1 + p = 18 + 2 = 20$ and $e = 20 + 1 = 21$. Thus the type becomes $[20 * 21; 9^9]$.

In both cases, if the type is one of these, then $g = 1, \omega = 10, k = 8$.

If $p = 3$ then $k = 9, \nu_1 = 10, k = 9, B = 1$. Then $\sigma = 2\nu_1 + p = 23$ and $e = 23 + 10 = 33$. Thus the type becomes $[23 * 33, 1; 10^9]$.

If the type of the pair (S, D) is this, then $g = 1, \omega = 11, k = 9$.

iii) Assume that $\rho = 3$. Then $k + 4 = 3\tilde{k} + 6$. Hence, $k - 2 = 3\tilde{k}$. From $\tilde{k} = p(k - 2p)$, it follows that

$$6p^2 - 2 = (3p - 1)k \geq 3p \cdot (3p - 1). \tag{14}$$

Thus

$$6p^2 > 6p^2 - 2 \geq 3p \cdot (3p - 1) = 9p^2 - 3p.$$

Hence, $3p > 3p^2$. This is a contradiction.

5.3 case when $g = 2$

Suppose that $g = 2$. Then

$$k(2i - 2 - \rho) + (i - 1)^2 = \rho(\tilde{k} + i - 3).$$

Moreover, since $\rho\nu_1 = i - 1 + 2k$ and $i \leq 2$, it follows that $\rho > 0$. Hence, $i = 2$.

Therefore, $k(2 - \rho) + 1 = \rho(\tilde{k} - 1)$. Thus $\rho = 1$ and so $\tilde{k} = k + 2$. Hence,

$$(p - 1)k = 2p^2 + 2.$$

Thus $p > 1$ and

$$2p + 2 + \frac{4}{p - 1} = k.$$

Hence, we obtain 1) $p = 2$,or 2) $p = 3$ or 3) $p = 5$.

1) $p = 2$. $k = 6 + 4 = 10$ and $B = 1, u = 2$. Hence, $\rho\nu_1 = i - 1 + 2k = 21$. We obtain three cases i) $\rho = 1, \nu_1 = 21$,ii) $\rho = 3, \nu_1 = 7$ iii) $\rho = 7, \nu_1 = 3$ to examine, separately.

i) $\rho = 1, \nu_1 = 21$. Then $\sigma = 44, e = 67$.
Thus the type becomes $[44 * 67, 1; 21^9]$.
If the type of the pair (S, D) is this, then $g = 2, Z^2 = 12, \omega = 12, k = 10$.

ii) $\rho = 3, \nu_1 = 7$. Then $\sigma = 16, e = 25$. Thus the type becomes $[16 * 25, 1; 7^{11}]$. However, if the type of the pair (S, D) is this, then $g = 9, \omega = 19, k = 10$.

iii) $\rho = 7, \nu_1 = 3$. Then $\sigma = 8, e = 213$. Thus the type becomes $[8 * 13, 1; 3^{15}]$. However, if the type of the pair (S, D) is this, then $g = 11, \omega = 21, k = 10$.
In the cases 2) and 3), it is easy to derive contradictions.

5.4 case when $g > 2$

Suppose that $\overline{g} \geq 2$, we obtain

$$k\nu_1 \leq -\tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g} \leq -\tilde{k} + (\nu_1 - 1)\omega - 2(\nu_1 - 3).$$

Hence,

$$(k + 2)\nu_1 \leq -\tilde{k} + \nu_1\omega + 6 - \omega$$

and so

$$k + 2 \leq -\tilde{k} + \omega + \frac{6 - \omega}{\nu_1}. \tag{15}$$

Since $\omega - k \leq 2$, it follows that $6 - \omega \geq 0$.
If $\omega = 6$ then $\tilde{k} = 0, k = 4$. Moreover, since $\overline{g} \geq 2$, it follows that

$$\widetilde{\mathcal{Z}} = \nu_1(2 - \overline{g}) - 6 + 3\overline{g} \leq 0.$$

Hence, $\overline{g} = 2, \widetilde{\mathcal{Z}} = 0$. Thus,

$$Y = r\nu_1 = 8\nu_1 + k + \omega_1 = 8\nu_1 + 4 + 6 - 2 = 8\nu_1 + 8,$$

and $\rho\nu_1 = 8$. Thus we have either i) $\rho = 1, \nu_1 = 8$ or ii) $\rho = 2, \nu_1 = 4$.

i) $\rho = 1, \nu_1 = 8$. Then $\sigma = 16, u = 2$. The type becomes $[16 * 18; 8^9]$.
If the type of the pair (S, D) is this , then $g = 3, \omega = 6, k = 4$.

ii) $\rho = 2, \nu_1 = 4$. Then $\sigma = 8, u = 2$. The type becomes $[8 * 10; 4^{10}]$.

If the type of the pair (S, D) is this , then $g = 3, \omega = 6, k = 4$.

If $4 \leq \omega \leq 5$ then by the inequality (15), we have $\tilde{k} = 0, k = 2u, \omega = k+2$.
Hence we get $\omega = 4, k = 2$. By $\rho\nu_1 = 2k + i - \overline{g} = 6 - \overline{g}, \overline{g} \geq 2$ we get $\overline{g} = 2, \rho\nu_1 = 4$. Hence, $\rho = 1, \nu_1 = 4, \sigma = 8, e = 9, r = 9$.

The type becomes $[8*9; 4^9]$. However,if the type of the pair (S, D) is this , then $g = 2, \omega = 3, k = 2, i = 1$.

If $\omega = 3$ then $\omega = k + i$; hence, $i = 1, k = 2$. By (15), we get $\nu_1 = 3, p = 0, \sigma = 6$.

6 case when $j_1 = 1, j_2 = 0$

Supposing that $j_1 = 1, j_2 = 0$, we obtain

$$\rho\nu_1 = i + 1 - \overline{g} + 2k. \tag{16}$$

Since $\nu_1 - 1 = \widetilde{\mathcal{Z}}$, it follows that

$$\nu_1 - 1 = \widetilde{\mathcal{Z}} \leq -k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g}.$$

Thus

$$(1 + k)\nu_1 \leq (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g} + 1.$$

6.1 case when $\overline{g} \geq 1$

Supposing that $\overline{g} \geq 1$, we get

$$(2 + k)\nu_1 \leq (\nu_1 - 1)\omega + 4 = \nu_1\omega - \omega + 4.$$

Hence, if $i \leq 2$, then $g = 1, \omega = 4, k = 2; i = 2, B = 0, u = 1, p = 0$. Thus,

$$\rho\nu_1 = i + 1 - \overline{g} + 2k = 2 + 2k = 6.$$

We have two cases i) $\rho = 1, \nu_1 = 6$ or ii) $\rho = 2, \nu_1 = 3$.

i) $\rho = 1, \nu_1 = 6$. Then we obtain $\sigma = 12, e = 13, r = 9$. Thus the type becomes $[12 * 13; 6^8, 5]$.

Conversely, if the type of the pair (S, D) is this , then $g = 2, \omega = 4, k = 2$.

ii) $\rho = 2, \nu_1 = 3$. we obtain $\sigma = 6, e = 7, r = 10$, and so the type becomes $[6 * 7; 3^9, 2]$. But $\sigma \geq 7$ was assumed.

6.2 case when $g = 0$

Assume $g = 0$. Then the formula (FEQ) turns out to be

$$k(2i - \rho) + (i + 1)^2 - 1 = \rho(\tilde{k} + i + 2).$$

6.2.1 case when $i = 1$

Suppose that $i = 1$. Then $k(2 - \rho) + 3 = \rho(\tilde{k} + 3)$. Hence $\rho = 1$. Thus $k = \tilde{k}$, which implies

$$2p^2 - 2 + 2 = (p - 1)k.$$

In other words,

$$2p + 2 + \frac{2}{p - 1} = k.$$

Hence, $p = 2$ or 3.

i) If $p = 2$ then $k = 8$ and $\nu_1 = 2k + 3$.

In the case when $B = 1$, we obtain $\nu_1 = 2k + 3 = 19, k = 8 = 3 \cdot 2 + 2u$; $u = 1$. Hence $\sigma = 38 + 2 = 40, e = 40 + 19 + 1 = 60$. Therefore, the type becomes $[40 * 60, 1; 19^8, 18]$.

Conversely, if the type of the pair (S, D) is this ,then $g = 0, \omega = 9, k = 8$.

In the case when $B = 0$, we obtain $\nu_1 = 2k + 3 = 19, k = 8 = 4 \cdot 2 + 2u$; $u = 0$. Hence $\sigma = 38 + 2 = 40, e = 40$. Therefore, the type becomes $[40 * 40; 19^8, 18]$. Conversely,if the pair has this type, then $g = 0, \omega = 9, k = 8$.

ii) If $p = 3$ then $k = 9$ and $\nu_1 = 2k + 3 = 21$. Thus $B = 1, u = 0$. $\sigma = 42+3 = 45, e = 45+21 = 66$. Therefore, the type becomes $[45*66, 1; 21^8, 20]$. Conversely, if the pair has this type ,then $g = 0, \omega = 10, k = 9, Z^2 = 8$.

6.3 case when $i = 2$

Suppose that $i = 2$. Then

$$k(4 - \rho) + 8 = \rho(\tilde{k} + 4) = \rho(p(k - 2p) + 4).$$

Further,

$$k(4 - \rho) + 8 - 4\rho = \rho pk - 2p^2\rho.$$

Hence,

$$2p^2\rho + 8 - 4\rho = k(\rho p + \rho - 4) \geq 3p(\rho p + \rho - 4) = 3p^2\rho + 3p\rho - 12p.$$

Therefore,

$$\frac{12p+8}{p^2+3p+4} \geq \rho.$$

Hence, $\rho \leq 2$.

i) Suppose that $\rho = 1$. Then

$$3k + 4 = pk - 2p^2.$$

Hence,

$$2p^2 + 4 = (p-3)k$$

and $p \geq 4$. Thus

$$2(p+3) + \frac{22}{p-3} = k.$$

Since $k \geq 3p$, we obtain the following tables.

Table 7: case when $\rho = 1, B = 1$

$p-3$	p	$2(p+3)$	$22/(p-3)$	k	ν_1	σ	e	$3p$	u
1	4	14	22	36	76	156	244	12	12
2	5	16	11	27	58	121	185	15	6

Conversely, if the type is $[156*244, 1; 76^8, 75]$,then $g = 0, \omega = 38, k = 36$. If the type is $[121*185; 58^8, 57]$,then $g = 0, \omega = 29, k = 27$.

Table 8: case when $\rho = 1, B = 0$

$p-3$	p	$2(p+3)$	$22/(p-3)$	k	ν_1	σ	e	$4p$	u
1	4	14	22	36	76	156	168	16	12
2	5	16	11	27	58	121	127	20	6

Conversely, if the type is $[156*168; 76^8, 75]$,then $g = 0, \omega = 38, k = 36$. If the type is $[121*127; 58^8, 57]$,then $g = 0, \omega = 29, k = 27$.

ii) Suppose that $\rho = 2$. Then $\nu_1 = k + 2$ and

$$2k + 8 = 2(\tilde{k} + 4) = 2(p(k - 2p) + 4).$$

Hence,

$$2p^2 = (p - 1)k$$

and then

$$2(p + 1) + \frac{2}{p - 1} = k.$$

If $p = 2$, then $k = 8$ and $\nu_1 = 10$ and $\sigma = 22$.

If $B = 1$ then $\sigma = 22, e = 33$. Therefore, the type becomes $[22 * 33, 1; 10^9, 9]$.

Conversely, if the pair has this type, then $g = 0, \omega = 10, k = 8$.

If $B = 0$ then $\sigma = 22, e = 22$. Therefore, the type becomes $[22*22; 10^9, 9]$.

Conversely, if the pair has this type, then $g = 0, \omega = 10, k = 8$.

If $p = 3$, then $k = 9$ and $\nu_1 = 11$ and $\sigma = 25, e = 36$. Therefore, the type becomes $[25 * 36, 1; 11^9, 10]$. Conversely,if the pair has this type, then $g = 0, \omega = 11, k = 9, Z^2 = 8$.

6.4 case when $g = 1$

Suppose that $g = 1$. Then

$$k(2i - 2 - \rho) + i^2 - 1 = \rho(\tilde{k} + i - 1),$$

and $i = 2$ and $\rho = 1$. Thus $\nu_1 = 3 + 2k$ and $k + 2 = \tilde{k}$. Accordingly,

$$2p + 2 + \frac{4}{p - 1} = k.$$

Hence, we obtain the following tables.

Table 9: case when $\rho = 1$ and $i = 2$; $B = 1$

$p-1$	p	$2(p+1)$	$4/(p-1)$	k	ν_1	σ	e	$3p$	u
1	2	6	4	10	23	48	73	6	2

Table 10: $\rho = 1$ and $i = 2$; $B = 0$

$p-1$	p	$2(p+1)$	$4/(p-1)$	k	ν_1	σ	e	$4p$	u
1	2	6	4	10	23	48	49	8	1

Conversely, if the pair has the type $[48 * 73, 1; 23^8, 22]$,then $g = 1, \omega = 12, k = 10$.

Conversely, if the pair has the type $[48 * 49; 23^8, 22]$,then $g = 1, \omega = 12, k = 10, Z^2 = 11$.

7 case when $j_1 = 2, j_2 = 0$

Supposing that $j_1 = 2, j_2 = 0$, we obtain

$$\rho\nu_1 = i + 1 - \overline{g} + 2k. \tag{17}$$

Since $2\nu_1 - 2 = \widetilde{\mathcal{Z}}$, it follows that

$$2\nu_1 - 2 \leq \widetilde{\mathcal{Z}} \leq -k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g}.$$

Thus

$$(2 + k)\nu_1 \leq (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g} + 2.$$

7.1 case when $\overline{g} \geq 0$

Supposing that $\overline{g} \geq 0$, we get

$$(2 + k)\nu_1 \leq (\nu_1 - 1)\omega + 2.$$

Hence, it follows that

$$2 + k \leq k + 2 + \frac{2 - \omega}{\nu_1}.$$

Therefore,since $i = \omega - k \leq 2$, we obtain $\omega = 2, g = 1, k = 0, B = 0$. Then

$$Y = (r - 2)\nu_1 + 2(\nu_1 - 1) = r\nu_1 - 2 = 8\nu_1 + \omega = 8\nu_1 + 2.$$

Thus $\rho\nu_1 = 4$, which implies $\rho = 1, \nu_1 = 4$.

Therefore, the type becomes $[8 * 8; 4^7, 3^2]$.

Conversely,if the pair has this type, then $g = 1, \omega = 2, k = 0$.

7.2 case when $g = 0$

The formula (FEQ) turns out to be

$$k(2i - 2 - \rho) + (i + 1)^2 - 4 = \rho(\tilde{k} + i + 1).$$

Since $i \leq 2$, it follows that $i = 2$. Hence, $\rho\nu_1 = 5 + 2k$ and

$$k(2 - \rho) + 5 = \rho(\tilde{k} + 3).$$

Hence, $\rho = 1$ and $k + 2 = \tilde{k} = p(k - 2p)$. Thus $2p^2 + 2 = (p - 1)k$; hence,

$$2p + 2 + \frac{4}{p-1} = k.$$

This induces $p = 2$ or 3 or 5.

But, if $p = 3$ then $k = 10, B = 1, u = 0$. By the way, $k = 3p + 2u = 9$, which contradicts $k = 10$.

Moreover, if $p = 5$ then $k = 10, B = 1, u = 0$. By the way, $k = 3p+2u = 9$, which contradicts $k = 10$.

Table 11: case when $B = 1$

$p-1$	p	$2p+2$	$4/(p-1)$	k	u	ν_1	σ	e	$3p+2u$
1	2	6	4	10	2	25	52	79	10

Consequently, the type becomes $[52 * 79, 1; 25^7, 24^2]$.

Table 12: case when $B = 0$

$p-1$	p	$2p+2$	$4/(p-1)$	k	u	ν_1	σ	e	$4p+2u$
1	2	6	4	10	1	25	52	53	10

Consequently, the type becomes $[52 * 53; 25^7, 24^2]$.
Conversely,if the type of (S, D) is this, then $g = 0, \omega = 12, Z^2 = 10$.

8 case when $j_1 = 0, j_2 = 1$ or $t_2 = 1$

Assume $j_1 = 0$ and either $j_2 = 1$ or $t_2 = 1$. Then $2\nu_1 - 4 = \widetilde{\mathcal{Z}}$ and hence,

$$2\nu_1 - 4 \leq -k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g}.$$

Thus

$$(2 + k)\nu_1 \leq (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g} + 4.$$

8.1 case when $g > 0$

Suppose that $\overline{g} \geq 0$. Then

$$(2 + k)\nu_1 \leq \nu_1\omega + 4 - \omega.$$

Hence, $\omega = 4, k = 2, g = 1$.

We have either i) $t_2 = 1$ or ii) $t_{\nu_1 - 2} = 1$.

i) $t_2 = 1$. Then by $Y = 8\nu_1 + 2 + \omega = (r - 1)\nu_1 + 2$, we get

$$(\rho - 1)\nu_1 = k + 2 = 4.$$

Hence, $\nu_1 = 4, r = 10$.

Consequently, the type becomes $[8 * 9; 4^9, 2]$.

Conversely, if the pair has this type , then $g = 1, \omega = 4, k = 2$.

ii) $t_{\nu_1 - 2} = 1, \nu_1 > 4$. Then by $Y = 8\nu_1 + 2 + \omega = r\nu_1 - 2$, we get

$$\rho\nu_1 = i + 2 - \overline{g} + 2k = k + 6 = 8.$$

Thus, we have $\rho = 1, \nu_1 = 8$. Then the type becomes $[16 * 17; 8^8, 6]$.

Conversely, if the pair has this type , then $g = 1, \omega = 4, k = 2, g = 1, Z^2 = 3$.

8.2 case when $g = 0$

Suppose that $\overline{g} = -1$. Then we have two cases a) $t_{\nu_1 - 2} = 1$ and b) $t_2 = 1$.

In the case a), the formula (FEQ) turns out to be

$$k(2i - 2 - \rho) + (i + 1)^2 - 16 = \rho(\tilde{k} + i - 1).$$

Since $i \leq 2$, it follows that $i = 2$ and that

$$k(2-\rho) + 9 - 16 = \rho(\tilde{k} + 1).$$

Hence, $\rho = 1$ and

$$2p^2 - 2 + 6 = (p-1)k \geq 3p^2 - 3p.$$

Therefore,

$$k = 2p + 2 + \frac{6}{p-1}.$$

Hence, we obtain the following table.

Table 13:

$p-1$	p	$2p+2$	$6/(2p+2)$	k	$\nu_1 = 2k+5$	u	σ	e
1	2	6	6	12	29	3	60	92
2	3	8	3	11	27	1	57	85
3	4	10	2	12	29	0	62	91

Consulting this table,if $p = 2$, then we obtain the following types:
$B = 1$. Then $u = 3$ and the type becomes $[60 * 92, 1; 29^8, 27]$.
Conversely, if the pair has this type , then $g = 0, \omega = 14, k = 12, Z^2 = 12$.
$B = 0$. Then $u = 2$ and the type becomes $[60 * 62; 29^8, 27]$.
Conversely, if the pair has this type , then $g = 0, \omega = 14, k = 12, Z^2 = 12$.

In the case b), we have $j_1 = j_2 = 0, t_2 = 1$ and that $\overline{g} = -1$.
By the way since $Y = \nu_1(r-1) + 2$ and $X = \nu_1^2(r-1) + 4$ we obtain

- $(\rho - 1)\nu_1 = k + \omega - \overline{g} - 2 = 2k + i - \overline{g} - 2$,
- $(\rho - 1)\nu_1^2 = -4 + 2k\nu_1 + \tilde{k} + \omega - 3\overline{g} = -4 + 2k\nu_1 + \tilde{k} + k + i - 3\overline{g}$.

By

$$(\rho - 1)\nu_1^2 = -4 + 2k\nu_1 + \tilde{k} + k + i - 3\overline{g} = 2k + i - \overline{g} - 2)\nu_1$$

we get

$$(i - \overline{g} - 2)\nu_1 = -4 + \tilde{k} + k + i - 3\overline{g}.$$

Thus,

$$(i - \overline{g} - 2)(2k + i - \overline{g} - 2) = (-4 + \tilde{k} + k + i - 3\overline{g})(\rho - 1).$$

Putting $\overline{g} = -1$, we get

$$(i-1)(2k+i-1) = (\tilde{k}+k+i-1)(\rho-1).$$

Since $i \leq 2$, it follows that $i = 2$ and therefore,

$$2k+1 = (\tilde{k}+k+1)(\rho-1) = (\tilde{k}+1) + (\rho-1) + k(\rho-1).$$

Then $\rho - 1 > 0$ and hence,

$$k(3-\rho) = (\tilde{k}+k+1)(\rho-1) - 1 > 0.$$

Thus, $\rho = 2$ and finally, we obtain $\tilde{k} = k$ and so $2p^2 - 2 + 2 = k$. Then we get either 1) $p = 2$; hence $k = 8 = 3 \times 2 + 2$ or $k = 8 = 4 \times 2$ or 2) $p = 3$; hence$k = 9$. By $\nu_1 = (\rho-1)\nu_1 = k + \omega - \overline{g} - 2 = 2k + 2 + 1 - 2 = 2k + 1$, we get

1). $k = 8, \nu_1 = 2k + 1 = 17$. If $B = 1$ then $u = 1$ else if $B = 0$ then $u = 0$. Then we have the following cases:

i) $B = 1$. Thus $\sigma = 2\nu_1 + p = 36, e = \sigma + u + \nu_1 = 54$ and the type becomes $[36 * 54, 1; 17^9, 2]$.

ii) $B = 0$. Thus the type becomes $[36 * 36; 17^9, 2]$.

Conversely, if the pair has this type , then $g = 0, \omega = 10, k = 8$.

2) $p = 3$. Then $k = 9, \nu_1 = 2k + 1 = 19, u = 0, B = 1$. Thus the type becomes $[41 * 60, 1; 19^9, 2]$.

Conversely, if the pair has this type , then $g = 0, \omega = 11, k = 9$.

9 case when $\widetilde{\mathcal{Z}} \geq 3(\nu_1 - 3)$

Suppose that $k > 0, \nu_1 \geq 3$ and $\widetilde{\mathcal{Z}} \geq 3(\nu_1 - 3)$.

From definition, it follows that

$$\begin{aligned} 3(\nu_1 - 3) \leq \widetilde{\mathcal{Z}} &= \nu_1 Y - X && (18)\\ &= -k\nu_1 + (\nu_1 - 1)\omega + (3 - \nu_1)\overline{g} - \tilde{k} && (19)\\ &\leq -k\nu_1 + (\nu_1 - 1)\omega - (3 - \nu_1) - \tilde{k} && (20)\\ & && (21) \end{aligned}$$

and that

$$(k+2)\nu_1 \le (\nu_1 - 1)\omega - \tilde{k} + 6.$$

Hence,

$$k + 2 \le \nu_1 \le \omega + \frac{6 - \omega - \tilde{k}}{\nu_1}. \tag{22}$$

Thus we have three cases 1) $6 - \omega - \tilde{k} < 0$, 2) $6 - \omega - \tilde{k} = 0$ and 3) $6 - \omega - \tilde{k} > 0$.

1) $6 - \omega - \tilde{k} < 0$. Then $k \le \omega - 3$. Hence, $i \ge 3$, which contradicts the hypothesis: $i \le 2$.

2) $6 - \omega - \tilde{k} = 0$. Then from the formula (22), it follows that $k + 2 \le \omega = k + i$. Hence, $i = 2$. Thus the formula (18) induces

$$k + 3 + \overline{g} \le \omega + \frac{3\overline{g} + 9 - \omega - \tilde{k}}{\nu_1}.$$

By

$$1 + \omega - 2 + \overline{g} = k + 3 + \overline{g} \le \omega + \frac{3\overline{g} + 9 - \omega - \tilde{k}}{\nu_1} = \omega + \frac{3g}{\nu_1},$$

we get

$$\nu_1 g \le 3g.$$

We have two cases I) $\nu_1 \ge 4$ and II) $\nu_1 = 3$ to examine ,separately.

9.1 case when $\nu_1 \ge 4$

I) $\nu_1 \ge 4$. Then $g = 0$. Since $6 = \omega - \tilde{k}$ and $\omega = k + 2$, it follows that $\tilde{k} + k = 4$. Then $k = 3$ or 4. To verify this we examine the following two cases:

If $\tilde{k} > 0$ then $p > 0$ and so $k \ge 3$. Hence, $\tilde{k} = 1$ and $k = 3$, which implies that $p = 1, B = 1, u = 0$.

If $\tilde{k} = 0$ then $p = 0$ and so $k = 2u = 4$.

By the way, from the next formulae :

- $Y = 8\nu_1 + k + \omega + 1 = 8\nu_1 + 2k + 3,$
- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega + 3 = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + k + 5,$

it follows that

$$\widetilde{\mathcal{Z}} = \nu_1 Y - X = 3\nu_1 - (\tilde{k} + k + 5) = 3\nu_1 - 9.$$

Letting $a = t_{\nu_1 - 1}, b = t_{\nu_1 - 2} + t_2, c = t_3 + t_{\nu_1 - 3}$, we obtain

$$a(\nu_1 - 1) + 2b(\nu_1 - 2) + 3c(\nu_1 - 3) = 3\nu_1 - 9.$$

If $c = 0$ then $(a + 2b - 3)\nu_1 = a + 4b - 9$. Since $\nu_1 \geq 3$, it follows that

$$3(a + 2b - 3) \leq a + 4b - 9.$$

Hence, $3a + 4b \leq a + 4b$, which implies $a = b = 0$. Therefore,$a = b = 0, c = 1$.

From $t_3 + t_{\nu_1 - 3} = 1$, it follows that i) $t_3 = 0, t_{\nu_1 - 3} = 1$ or ii) $t_3 = 1, t_{\nu_1 - 3} = 0$.

Hence, we have either i) $Y = \nu_1(r - 1) + \nu_1 - 3 = \nu_1 r - 3$ or ii) $Y = \nu_1(r - 1) + 3$.

i) $Y = \nu_1 r - 3$. Then by making use of $Y = 8\nu_1 + 2k + 3$ we get $\rho\nu_1 = 2k + 6$, where $\rho = r - 8$.

Then we have the following cases to examine, separately.

1) $k = 4$. Then $p = 0, B = 0, \omega_1 = 7$. Hence, $\rho\nu_1 = 14$. Thus $\nu_1 = 14$ or 7, for $\nu_1 > 3$ by hypothesis.

If $\nu_1 = 14$ then $\rho = 1, r = 9, \sigma = 28, e = \sigma + u = 30$. The type turns out to be $[28 * 30; 14^8, 11]$. If the pair has this type ,then $g = 0, \omega = 6, k = 4$.

If $\nu_1 = 7$ then $\rho = 2, r = 10, \sigma = 14, e = \sigma + u = 16$. The type turns out to be $[14 * 16; 7^9, 4]$. If the pair has this type ,then $g = 0, \omega = 6, k = 4$.

2) $k = 3$. Then $p = 1, B = 1, u = 0, \omega_1 = 7$. Hence, $\rho\nu_1 = 12$. Thus $\nu_1 = 12$ or 6, for $\nu_1 > 3$ by hypothesis. Hence,

If $\nu_1 = 12$ then $\rho = 1, r = 9, \sigma = 25, e = \sigma + u + \nu_1 = 37$. The type turns out to be $[25 * 37, 1; 12^8, 9]$. If the pair has this type ,then $g = 0, \omega = 5, k = 3$.

If $\nu_1 = 6$ then $\rho = 2, r = 10, \sigma = 13, e = \sigma + u + \nu_1 = 19$. The type turns out to be $[13 * 19, 1; 6^9, 3]$. If the pair has this type ,then $g = 0, \omega = 5, k = 3$.

ii) $Y = \nu_1(r - 1) + 3 = 2k + 3$. We obtain $(\rho - 1)\nu_1 = 2k$.

1) $k = 4$. Then $p = 0, u = 2$. Hence, $(\rho - 1)\nu_1 = 8$. Thus $\nu_1 = 8$ or 4, for $\nu_1 > 3$ by hypothesis.

If $\nu_1 = 8$ then $\rho = 2, r = 10, \sigma = 16, e = \sigma + u = 18$. The type turns out to be $[16 * 18; 8^9, 3]$. If the pair has this type ,then $g = 0, \omega = 6, k = 4$.

If $\nu_1 = 4$ then $\rho = 3, r = 11, \sigma = 8, e = \sigma + u = 10$. The type turns out to be $[8 * 10; 4^{10}, 3]$. If the pair has this type ,then $g = 0, \omega = 6, k = 4$.

2) $k = 3$. Then $p = 1, B = 1, u = 0, \omega_1 = 6$. Hence, $(\rho - 1)\nu_1 = 6$. Thus $\nu_1 = 6$ for $\nu_1 > 3$ by hypothesis. Hence,

If $\nu_1 = 6$ then $\rho = 2, r = 10, \sigma = 12, e = \sigma + u + \nu_1 = 18$. The type turns out to be $[13 * 19, 1; 6^9, 3]$. If the pair has this type ,then $g = 0, Z^2 = 2, \omega = 5, k = 3$.

9.2 case when $\nu_1 = 3$

II) $\nu_1 = 3$. Then $\widetilde{\mathcal{Z}} = 2t_2$ and since $\omega = k + i$, where $i \leq 2$, it follows that

$$\widetilde{\mathcal{Z}} = \nu_1(i - \overline{g}) - (k + i + \tilde{k} - 3\overline{g}).$$

Hence, $2i = k + \tilde{k} + 2t_2 \leq 4$.

However by $\sigma = 6 + p \geq 7$ by hypothesis, we get $p > 0$.

Hence, $\tilde{k} = 1, k = 3, p = 1, u = 0, B = 1$. Moreover , $i = 2, t_2 = 0$. Thus, the type becomes $[7 * 10, 1; 3^r]$ where $33 - 3r \geq 0$.

Conversely, if the pair has this type , then $\overline{g} = 32 - 3r, 2\omega = (\sigma - 3)\widetilde{B} - 6\sigma = 10$. Hence, $\omega = 5$, which is equal to $k + 2$.

10 case when $\widetilde{\mathcal{Z}} < 3(\nu_1 - 3)$

In the case when $\widetilde{\mathcal{Z}} < 3(\nu_1 - 3)$, we obtain

$$a(\nu_1 - 1) + 2b(\nu_1 - 2) < \widetilde{\mathcal{Z}} < 3(\nu_1 - 3).$$

Here $a = t_{\nu_1 - 1}, b = t_{\nu_1 - 2} + t_2$.

Then we have either i) $a = 1, 2$ and $b = 0$ or ii) $a = 0$ and $b = 1$.

But these cases were discussed before.

11 estimate of genus in terms of ω

From the fundamental equalities, we obtain

$$0 \leq \widetilde{\mathcal{Z}} = B_2(\nu_1 - \sigma)\sigma - k\nu_1 - \tilde{k} + (\nu_1 - 1)\omega - (\nu_1 - 3)\overline{g}.$$

Thus

$$B_2(\sigma - \nu_1)\sigma + (\nu_1 - 3)\overline{g} + k\nu_1 + \tilde{k} \leq -\widetilde{\mathcal{Z}} + (\nu_1 - 1)\omega. \tag{23}$$

In particular,

$$(\nu_1 - 3)\overline{g} + k\nu_1 + \tilde{k} \leq (\nu_1 - 1)\omega. \tag{24}$$

Assuming $\nu_1 \geq 4$, we get the following

Theorem 3

$$\overline{g} \leq \frac{\nu_1 - 1}{\nu_1 - 3}\omega. \tag{25}$$

Moreover, if $\overline{g} = \frac{\nu_1 - 1}{\nu_1 - 3}\omega$ *then the type becomes* $[2\nu_1 * 2\nu_1; \nu_1^r], r = 1, 2, \cdots, 7$ *and their associates:*

Hence, the following estimate is obtained.

Corollary 1 *If* $\nu_1 \geq 4$ *, then*

$$\overline{g} \leq 3\omega.$$

Moreover, if $\overline{g} = 3\omega$ *then* $\nu_1 = 4$ *and the type becomes* $[8 * 8; 4^r], r = 1, 2, \cdots, 7$ *and their associates:*

12 another estimate

For a positive integer $n \geq 4$, define $\widetilde{F}(n)$ to be $(n-1)\omega - (n-3)\overline{g}$ and $\widetilde{F}(n)_0$ to be $(n-1)\omega_0 - (n-3)\overline{g}_0$, where $\overline{g}_0 = \frac{(C+K_0)\cdot C}{2}, \omega_0 = \frac{(C+3K_0)\cdot C}{2}$. Then

$$\widetilde{F}(n) - \widetilde{F}(n)_0 = \sum_{j=2}^{\nu_1}(n-j)jt_j.$$

To verify the above, we notice

$$\widetilde{F}(n) = \frac{1}{2}(n-1)(D + 3K_S) \cdot D - \frac{1}{2}(n-3)(D + K_S) \cdot D = (D + nK_S) \cdot D.$$

and

$$\widetilde{F}(n)_0 = (C + nK_0) \cdot C = (\sigma - n)\widetilde{B} - 2n\sigma.$$

As a matter of fact,

$$C^2 = \widetilde{B} \cdot C, K_0 \cdot C = \omega_0 - \overline{g}_0 = \tau_3/2 - 9 - \tau_1/2 + 1 = -2\sigma - \widetilde{B}.$$

Furthermore,

$$((D + nK_S) - (C + nK_0)) \cdot (D - C) = \sum_{j=2}^{\nu_1} (n - j)jt_j.$$

Here, $\widetilde{B} = 2e - B\sigma$ and $\tau_m = (\sigma - m)(\widetilde{B} - 2m)$.

Then defining $\widetilde{\mathcal{Z}}(n)$ to be $\sum_{j=2}^{\nu_1}(n - j)jt_j$, we obtain

$$\widetilde{F}(n) = \widetilde{F}(n)_0 + \widetilde{\mathcal{Z}}(n). \tag{26}$$

By Theorem 3, if $n \leq \nu_1$ then $\frac{n-1}{n-3}\omega \geq \frac{\nu_1-1}{\nu_1-3}\omega \geq \overline{g}$. Hence, $\widetilde{F}(n) \geq 0$.

Thus if $\widetilde{F}(n) < 0$, then $n > \nu_1$ and so $\widetilde{\mathcal{Z}}(n) > 0$.

12.1 computation of $\widetilde{F}(n)_0$

When $B \neq 1$, we get $\widetilde{B} = 2e - B\sigma = (B_2 + 2)\sigma + 2u$.

$$\begin{aligned}
\widetilde{F}(n)_0 &= (\sigma - n)\widetilde{B} - 2n\sigma \\
&= (\sigma - n)((B_2 + 2)\sigma + 2u) - 2n\sigma \\
&= (\sigma - n)B_2 + 2u(\sigma - n) + 2(\sigma - 2n)\sigma.
\end{aligned}$$

Thus if $\sigma \geq 2n$ then $\widetilde{F}(n)_0 \geq 0$.

To study the case when $\sigma < 2n$, we replace σ by $2n - j$ and get $\widetilde{F}(n)_0 = (n - j)B_2 + 2(n - j)u + j(j - 2n)$.

Hence, if $n > j$ and $\widetilde{F}(n)_0 < 0$ then $(n-j)B_2+2(n-j)u+j(j-2n) < 0$. This implies that

$$u < \frac{j(2n - j) - (n - j)B_2}{2(n - j)}.$$

Thus u is bounded for n.

When $B = 1$, we get $\widetilde{B} = \sigma + 2u + 2\nu_1$.
Replacing σ by $3n - j - 2$, we obtain

$$\begin{aligned}
\widetilde{F}(n)_0 &= (\sigma - n)(\sigma + 2u + 2\nu_1) - 2n\sigma \\
&= (2n - j - 2)(\sigma + 2u + 2\nu_1) - 2n\sigma \\
&= -(j + 2)\sigma + (2n - j - 2)(2u + 2\nu_1) \\
&= (2n - j - 2)(2u + 2\nu_1 - j - 2) - n(j + 2).
\end{aligned}$$

If $j = 0$ then

$$\widetilde{F}(n)_0 = (2n - 2)(2u + 2\nu_1 - 2) - 2n \geq 2(n - 2) \geq 4,$$

for $2u + 2\nu_1 \geq 4$ and $n \geq 4$.
If $j = 1$ and then $u + \nu_1 \geq 3$ then

$$\widetilde{F}(n)_0 = (2n - 3)(2u + 2\nu_1 - 3) - 3n \geq 3(n - 3) \geq 3.$$

If $j = 2$ and then $u + \nu_1 \geq 4$ then

$$\widetilde{F}(n)_0 = (2n - 4)(2u + 2\nu_1 - 4) - 4n \geq 4(n - 4) \geq 0.$$

Moreover,supposing that $2n - j - 2 > 0$, if $\widetilde{F}(n)_0 < 0$ then from $(2n - j - 2)(2u + 2\nu_1 - j - 2) - n(j + 2) < 0$ we get

$$2u < \frac{n(j+2)}{2n - j - 2} - 2\nu_1 + j + 2 \leq \frac{n(j+2)}{2n - j - 2} + j + 2.$$

However, if $\nu_1 = 1$ then $\widetilde{B} = \sigma + 2u$ and $u \geq 2$.
In this case, $g_0 = g = (\sigma - 1)(\sigma + 2u - 2)/2$, $\omega = (\sigma - 3)(\sigma + 2u)/2 - 3\sigma$.
Hence, $\widetilde{F}(n)_0 = (\sigma - n)(\sigma + 2u) - 2n\sigma = \sigma(\sigma - 3n) + 2u(\sigma - n)$.

12.2 case when $n = 4$

Assume that $n = 4$. Then $\widetilde{F}(4) = \widetilde{F}(4)_0 + \widetilde{Z}(4)$.

If $\sigma = 7$, then $\widetilde{F}(4)_0 = -35 + 6u$. Assuming $\widetilde{F}(4)_0 < 0$, we obtain $u = 2, 3, 5$.
If $\sigma = 8$, then $\widetilde{F}(4)_0 = 8(u - 3)$. Assuming $\widetilde{F}(4)_0 < 0$, we obtain $u = 2$.
If $\sigma = 9$, then $\widetilde{F}(4)_0 = -27 + 10u$. Assuming $\widetilde{F}(4)_0 < 0$, we obtain $u = 2$.

12.3 case in which $3\omega < \overline{g}$

Assume that $\sigma \geq 7$.
If $n = 4$ and $\nu_1 < 4$ then

$$\widetilde{F}(4) = 3\omega - \overline{g} = (\sigma - 4)\widetilde{B} - 8\sigma + 4t_2 + 3t_3$$

If $B = 1$ then $\widetilde{B} = \sigma + 2u + 2\nu_1$ and then $\sigma = 7$ or 8,9.

When $\sigma = 7$, $\widetilde{F}(4)_0 = 3(7 + 2u + 2\nu_1) - 56 = -35 + 6(u + \nu_1)$.
If $\nu_1 = 3$, then $\widetilde{F}(4) = -17 + 6u + 4t_2 + 3t_3 < 0$.
If $\nu_1 = 2$, then $\widetilde{F}(4) = -23 + 6u + 4t_2 < 0$.
If $\nu_1 = 1$, then $\widetilde{F}(4) = -35 + 6u < 0$. Hence,$2 \geq u \geq 5$. And $\omega = 17 + 4u, \overline{g} = 14 + 6u$.

When $\sigma = 8$, $\widetilde{F}(4)_0 = 4(8 + 2u + 2\nu_1) - 64 = -32 + 8(u + \nu_1)$.
If $\nu_1 = 3$, then $\widetilde{F}(4) = -8 + 8u + 4t_2 + 3t_3 < 0$.
If $\nu_1 = 2$, then $\widetilde{F}(4) = -16 + 8u + 4t_2 < 0$.
If $\nu_1 = 1$, then $\widetilde{F}(4) = -32 + 8u < 0$. Hence,$2 \geq u \geq 3$. And $\omega = 21 + 4u, \overline{g} = 14 + 6u$.

When $\sigma = 9$, $\widetilde{F}(4)_0 = 5(9 + 2u + 2\nu_1) - 72 = -27 + 10(u + \nu_1)$.
If $\nu_1 = 3$, then $\widetilde{F}(4) = -7 + 10u + 4t_2 + 3t_3 < 0$.
If $\nu_1 = 2$, then $\widetilde{F}(4) = -17 + 10u + 4t_2 < 0$.

When $\sigma = 10$, $\widetilde{F}(4)_0 = 6(10 + 2u + 2\nu_1) - 80 = -20 + 12(u + \nu_1) > 0$.

If $n = 5$ and $\nu_1 < 5$ then

$$\widetilde{F}(5) = 2(2\omega - \overline{g}) = (\sigma - 5)\widetilde{B} - 10\sigma + 6t_2 + 6t_3 + 4t_4.$$

If $n = 6$ and $\nu_1 < 6$ then

$$\widetilde{F}(6) = 5\omega - 3\overline{g} = (\sigma - 6)\widetilde{B} - 12\sigma + 8t_2 + 9t_3 + 8t_4 + 5t_5.$$

If $B \neq 1$ then $\widetilde{B} = 2e - B\sigma = B_2\sigma + 2u + 2\sigma$. Thus,for $\sigma = 2n$, it follows that

$$\widetilde{F}(n)_0 = (2n - n)(2B_2n + 2u + 4n) - 4n^2 \geq 0.$$

If $B = 1$ then $\widetilde{B} = \sigma + 2(u + \nu_1)$.

For $\sigma = 3n - 2$, it follows that

$$\widetilde{F}(n)_0 = (2n-2)(3n-2+2u+2\nu_1) - 2n(3n-2) \geq n-2.$$

Table 14: the types when $\widetilde{F}(n) < 0, n = 4$

$n = 4$

σ	B	ν_1	$\widetilde{F}(n)_0$	$\widetilde{\mathcal{Z}}(n)$
7	1	3	$-17+6u$	$4t_2+3t_3$
7	1	2	$-23+6u$	$4t_2$
7	0	2,3	$-14+6u$	$4t_2+3t_3$
8	1	3	$-26+8u$	$4t_2+3t_3$
8	1	2	$-28+8u$	$4t_2$
9	1	2	$-7+10u$	$4t_2$

Table 15: the types when $4 \leq \overline{g}/\omega$

u	σ	ν_1	t_2	t_3	ω	g	$\overline{g}$	$F(n)$	$\overline{g}/\omega$	fraction
2	7	0	0	0	1	27	26	−23	26	26
0	7	2	1	0	2	26	25	−19	12.5	$12\frac{1}{2}$
0	7	2	2	0	3	25	24	−15	8	8
3	7	0	0	0	5	33	32	−17	6.4	$6\frac{2}{5}$
0	7	3	0	1	5	30	29	−14	5.8	$5\frac{4}{5}$
0	7	2	3	0	4	24	23	−11	5.75	$5\frac{3}{4}$
2	8	0	0	0	6	35	34	−16	5.66	$5\frac{2}{3}$
0	7	3	0	2	5	27	26	−11	5.2	$5\frac{1}{5}$
1	7	2	1	0	6	32	31	−13	5.166666667	$5\frac{1}{6}$
0	7	3	1	1	6	29	28	−10	4.666666667	$4\frac{2}{3}$
0	7	3	0	3	5	24	23	−8	4.6	$4\frac{3}{5}$
0	7	2	4	0	5	23	22	−7	4.4	$4\frac{2}{5}$
1	7	2	2	0	7	31	30	−9	4.285714286	$4\frac{2}{7}$
4	7	0	0	0	9	39	38	−11	4.22222	$4\frac{2}{9}$
0	7	3	1	2	6	26	25	−7	4.166666667	$4\frac{1}{6}$
0	7	3	0	4	5	21	20	−5	4	4

Table 16: the types when $3 < \overline{g}/\omega < 4$

u	σ	ν_1	t_2	t_3	ω	g	$\overline{g}$	$F(n)$	$\overline{g}/\omega$	fraction
1	7	3	0	1	9	36	35	-8	3.888888889	$3\frac{8}{9}$
0	7	3	2	1	7	28	27	-6	3.857142857	$3\frac{6}{7}$
2	7	2	1	0	10	38	37	-7	3.7	$3\frac{7}{10}$
0	7	3	1	3	6	23	22	-4	3.666666667	$3\frac{2}{3}$
1	7	2	3	0	8	30	29	-5	3.625	$3\frac{5}{8}$
2	9	0	0	0	12	44	43	-7	3.5833	$3\frac{7}{12}$
1	7	3	0	2	9	33	32	-5	3.555555556	$3\frac{5}{9}$
0	7	2	5	0	6	22	21	-3	3.5	$3\frac{1}{2}$
0	8	3	0	1	11	39	38	-5	3.454545455	$3\frac{5}{11}$
0	7	3	2	2	7	25	24	-3	3.428571429	$3\frac{3}{7}$
1	7	3	1	1	10	35	34	-4	3.4	$3\frac{2}{5}$
5	7	0	0	0	13	45	44	-5	3.3846	$3\frac{5}{13}$
2	7	2	2	0	11	37	36	-3	3.272727273	$3\frac{3}{11}$
0	7	3	3	1	8	27	26	-2	3.25	$3\frac{1}{4}$
1	7	3	0	3	9	30	29	-2	3.222222222	$3\frac{2}{9}$
0	8	3	0	2	11	36	35	-2	3.181818182	$3\frac{3}{11}$
0	7	3	1	4	6	20	19	-1	3.166666667	$3\frac{1}{6}$
1	7	2	4	0	9	29	28	-1	3.111111111	$3\frac{1}{9}$
1	7	3	1	2	10	32	31	-1	3.1	$3\frac{1}{10}$
0	8	3	1	1	12	38	37	-1	3.083333333	$3\frac{1}{12}$
3	7	2	1	0	14	44	43	-1	3.071428571	$3\frac{1}{14}$

Table 17: the types when D are nonsingular plane curves

d	ω	g	$\overline{g}$	$\overline{g}/\omega$	fraction
9	0	28	27	∞	∞
10	5	36	35	7	7
11	11	45	44	4	4
12	18	49	48	2.66666	$\frac{8}{3}$

Table 18: the types when $\widetilde{F}(n) < 0$

$n = 5$				
σ	B	ν_1	$\widetilde{F}(n)_0$	$\widetilde{\mathcal{Z}}(n)$
7	1	3	$-44 + 4u$	$6t_2 + 6t_3$
7	1	2	$-48 + 4u$	$6t_2$
7	0	2,3	$-42 + 4u$	$6t_2 + 6t_3$
8	1	4	$-32 + 6u$	$6t_2 + 6t_3 + 4t_4$
8	1	3	$-38 + 6u$	$6t_2 + 6t_3$
8	1	2	$-44 + 6u$	$6t_2$
8	0	2,3,4	$-16 + 6u$	$6t_2 + 6t_3 + 4t_4$
9	1	4	$-22 + 8u$	$6t_2 + 6t_3 + 4t_4$
9	1	3	$-30 + 8u$	$6t_2 + 6t_3$
9	1	2	$-38 + 8u$	$6t_2$
9	0	4	$-18 + 8u$	$6t_2 + 6t_3 + 4t_4$
10	1	4	$-10 + 10u$	$6t_2 + 6t_3 + 4t_4$
10	1	3	$-20 + 10u$	$6t_2 + 6t_3$
10	1	2	$-30 + 10u$	$6t_2$
11	1	3	$-8 + 12u$	$6t_2 + 6t_3$
11	1	2	$-20 + 12u$	$6t_2$
12	1	2	$-8 + 14u$	$6t_2$

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